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NUMERICAL KNOT INVARIANTS OF FINITE TYPE FROM CHERN-SIMONS PERTURBATION THEORY

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ABSTRACT

Chern-Simons gauge theory for compact semisimple groups is analyzed from a perturbation theory point of view. The general form of the perturbative series expansion of a Wilson line is presented in terms of the Casimir operators of the gauge group. From this expansion new numerical knot invariants are obtained. These knot invariants turn out to be of finite type (Vassiliev invariants), and to possess an integral representation. Using known results about Jones, HOMFLY, Kauffman and Akutsu-Wadati polynomial invariants these new knot invariants are computed up to type six for all prime knots up to six crossings. Our results suggest that these knot invariants can be normalized in such a way that they are integer-valued.

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1. Introduction

Chern-Simons gauge theory [1] has been studied using non-perturbative as well as perturbative methods. A variety of non-perturbative studies have been carried out [2-10], which has led to many exact results related to polynomial invariants for knots and links. These include, on the one hand, a general approach to compute observables related to knots, links and graphs [11,12], and, on the other hand, more explicit applications as the computation of invariants for torus knots and links for the fundamental representation of the group $SU(N)$ [13] and for arbitrary representations of $SU(2)$ [14], the development of skein rules in a variety of situations [15,16,17], and the computation of invariants in more general cases [18,19]. All these studies cover the analysis of Jones [20,21], HOMFLY [22,21] and Kauffman [23] polynomials as well as Akutsu-Wadati polynomials [24] for some sets of knots and links. Perturbative studies of Chern-Simons gauge theory [26-42] have provided a rich amount of knowledge on the series expansion corresponding to Wilson lines. Many of the works in this respect deal with the problem of finding which is the renormalization scheme that leads to the exact results. In this paper we will not address this issue. We will assume that there exist a scheme in which the quantum corrections to the two and three-point functions account for the shift obtained in [1] of the Chern-Simons parameter k. The existence of this scheme has been proved to one loop [25,28,32,38] and to two loops [39]. In this paper we will concentrate on the structure of the perturbative series expansion. We present a procedure to define numerical knot invariants from the perturbative expansion. These kinds of studies were first made by the pioneering work [27].

In a previous paper [43] we analyzed Chern-Simons gauge theory from a perturbative point of view. The aim of this paper is to push forward that analysis to construct new numerical knot invariants. The main result obtained in [43] was the identification of all the Feynman diagrams of the pertubative series expansion of the vacuum expectation value of a Wilson line which contribute to its framing dependence. It was shown that the contribution from all those diagrams factorizes in the form predicted by Witten [1]. In this paper we attempt to organize the rest of the perturbative contributions in such a way that an infinite sequence of numerical knot invariants will be attached to a given knot.

A Feynman diagram associated to the vacuum expectation value of a Wilson line in Chern-Simons gauge theory provides a contribution which is the product of two factors times a power of the coupling constant $g \sim 1/\sqrt{k}$. The power of this constant characterizes the order of the Feynman diagram. One of the two factors depends on the gauge group and the representation chosen for the Wilson line. Its form is dictated by the Feynman rules. We will denote this factor as group factor. The second factor corresponds to a series of line and three-dimensional integrals of a certain integrand also dictated by the Feynman rules. We will denote this factor as the geometrical factor. The important point to remark is that given a Feynman diagram the group factor is independent of the closed curve corresponding to the Wilson line. On the other hand, the geometrical factor is independent of the group and representation chosen.

The idea behind the construction of the numerical knot invariants presented in this paper is the following. Let us consider all Feynman diagrams associated to the vacuum expectation value of a Wilson line at a given order in perturbation theory, except those which contribute to the framing dependence. These diagrams provide a set of group factors. Among these group factors there is a set of independent ones, *i.e.*, all the rest can be written as linear combinations of the ones in this set. The perturbative contribution at the order considered can be written as a sum of a series of numerical factors times the independent group factors. Since the whole contribution at a given order is a topological invariant and the group factors are chosen to be the independent ones, the numerical factors which enter the contribution are numerical invariants associated to the knot. These numerical knot invariants can be regarded as the independent geometrical factors. The number of independent group factors at each order in perturbation theory is finite. Therefore, these procedure allows to assign to each knot a finite set of numerical invariants at each order. This allows to associate a numerical sequence to each knot.

An important feature of these numerical knot invariants is that they are invariants of finite type or Vassiliev invariants [44,45,46]. This will be shown using recent results on the connection between polynomial knot invariants and Vassiliev invariants by Bar-Natan [47] and by Birman and Lin [48,49]. An important object associated to a Vassiliev invariant is its actuality table [44,45,48,49]. At a given order there are several invariants which have as type their order in perturbation theory. These invariants generate sets of actuality tables. There can not be more independent tables than the dimension of the space of Vassiliev invariants of the given type. To the order studied this is consistent with our results. Another important feature of the numerical knot invariants we are dealing with is that there seems to exist a normalization such that these knot invariants are integer-valued.

The new numerical knot invariants presented in this paper are framing independent, possess integral expressions, and are integers when properly normalized. Their integral expressions are rather cumbersome and therefore, in general, they are hard to compute. There is, however, an alternative way to obtain these invariants using exact results for knot polynomial invariants. Since the numerical invariants are universal in the sense that they are independent of the group and representation chosen, one may obtain sets of linear equations for them comparing the perturbative series expansion dictated by Chern-Simons gauge theory to known exact results. We apply this method in this work and we present the computation of the new numerical invariants for all prime knots up to six crossings to order six using known polynomial invariants. The results are summarized in Table I.

The paper is organized as follows. In sect. 2 we review the results of [43] and we summarize the Feynman rules of the theory. In sect. 3 the general form of the perturbative series expansion is given and the complete details are worked out up to order six. In sect. 4 the numerical knot invariants are defined and their features are analyzed; in particular, it is proven that they are of finite type. In sect. 5 these invariants are computed up to order six for all prime knots up to six crossings. Finally, in sect. 6 we state our final remarks. There are in addition three appendices. Appendix A contains our group theoretical conventions and the description of the calculation of Casimirs. Appendix B deals with the discussion of some technical details regarding the analysis of the general structure of the perturbative series expansion. In appendix C we present a summary of known polynomial invariants which are used in the calculations carried out in sect. 5.

2. Perturbative Chern-Simons gauge theory, Feynman rules and factorization theorem

In this section we will present first a brief review of Chern-Simons gauge theory from a perturbation theory point of view for a general compact semisimple group G. Our approach uses standard perturbative quantum field theory, and utilizes Feynman diagrams as the main tool. We apologize if this brief review occasionally becomes too explicit for a field theorist but we expect that the details will become useful for people more mathematically oriented.

Let us consider a G gauge connection A on a compact boundaryless three-dimensional manifold $\mathcal{M}$. The Chern-Simons action is defined as,

$$S_0(A) = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{2.1}$$

where k is an arbitrary positive integer. The symbol "Tr" denotes the trace in the fundamental representation of G. Notice that the action (2.1) does not depend on the metric on $\mathcal{M}$. In defining the theory from a perturbation theory point of view we must give a meaning to vacuum expectation values of operators, *i.e.*, to quantities of the form,

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [DA] \mathcal{O}(A) \exp\Big(iS(A)\Big), \tag{2.2}$$

where Z is the partition function,

$$Z = \int [DA] \exp\Big(iS(A)\Big). \tag{2.3}$$

In (2.2) $\mathcal{O}(A)$ is a function of A which might be local or non-local. The integrals entering (2.2) and (2.3) are functional integrals. We do not aim to a rigorous definition of these objects in terms of a measure, but to a exposition of the perturbative analysis of Chern-Simons theory. In this context these formal definitions are accurate enough. In order to obtain topological invariants, the operators entering (2.2) are chosen to be gauge invariant operators which do not depend on the three-dimensional metric. These operators are related to knots, links and graphs [1,50]. The requirement of gauge invariance comes from the fact that the exponential in (2.2) is invariant under gauge transformations of the form,

$$A_\mu \rightarrow h^{-1} A_\mu h + h^{-1}\partial_\mu h, \tag{2.4}$$

where h is an arbitrary continuous map $h : \mathcal{M} \rightarrow G$. This implies that the integration over gauge connections has to be restricted to an integration over gauge connections modulo gauge transformations. Our choice of gauge fixing will be the same as the one taken in [28,43].

Let us redefine the constant k and the field A in such a way that the action (2.1) becomes standard from a perturbation theory point of view. Defining

$$g = \sqrt{\frac{4\pi}{k}}, \tag{2.5}$$

one finds, after rescaling the gauge connection,

$$A_\mu \rightarrow g A_\mu, \tag{2.6}$$

that the Chern-Simons action takes the form:

$$S(A) = \int_{\mathcal{M}} \mathrm{Tr}(A \wedge dA + \frac{2}{3} g A \wedge A \wedge A). \tag{2.7}$$

If we choose a trivialization for the tangent bundle of the three manifold $\mathcal{M}$, the previous action can be written in components. Although the tangent bundle to $\mathcal{M}$ can be trivialized, this can be done in infinitely many ways. This is the origin of a phase ambiguity in Z which is discussed, for example, in [1], but this problem is immaterial for us. Following the group-theoretical conventions stated in Appendix A, the action in components reads,

$$S(A) = \frac{1}{2}\int\limits_{\mathcal{M}} \epsilon^{\mu\nu\rho}\Big(A^a_\mu \partial_\nu A^a_\rho - \frac{g}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\Big). \tag{2.8}$$

The standard procedure to compute (2.2) and (2.3) from a perturbation theory point of view involves the introduction of a source function J for the gauge field A and a modification of the action in the form,

$$S(A,J) = \int\limits_{\mathcal{M}} \Big[\frac{1}{2}\epsilon^{\mu\nu\rho}\Big(A^a_\mu \partial_\nu A^a_\rho - \frac{g}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho\Big) + J^{\mu\ a} A^a_\mu\Big]. \tag{2.9}$$

The original vacuum expectation value (2.2) and the partition function (2.3) are recovered setting $J = 0$. Standard arguments in quantum field theory allow to write,

$$\langle \mathcal{O} \rangle = \langle \mathcal{O}(J) \rangle \Big|_{J=0}, \tag{2.10}$$

where,

$$\begin{aligned} \langle \mathcal{O}(J) \rangle &= \frac{1}{Z[J]} \int [DA] \exp(iS(A,J)) \\ &= \frac{\mathcal{O}(\frac{\delta}{\delta J}) \exp\Big\{\frac{-g}{6}\int_{\mathcal{M}} \epsilon^{\mu\nu\rho} f^{abc} \frac{\delta}{\delta J^a_\mu}\frac{\delta}{\delta J^b_\nu}\frac{\delta}{\delta J^c_\rho}\Big\} \exp\Big\{\frac{i}{2}\int_{\mathcal{M}}\int_{\mathcal{M}} J^d_\sigma D^{\sigma\tau}_{de} J^e_\tau\Big\}}{\exp\Big\{\frac{-g}{6}\int_{\mathcal{M}} \epsilon^{\mu\nu\rho} f^{abc} \frac{\delta}{\delta J^a_\mu}\frac{\delta}{\delta J^b_\nu}\frac{\delta}{\delta J^c_\rho}\Big\} \exp\Big\{\frac{i}{2}\int_{\mathcal{M}}\int_{\mathcal{M}} J^d_\sigma D^{\sigma\tau}_{de} J^e_\tau\Big\}}. \end{aligned} \tag{2.11}$$

In this last expression $D^{\mu\nu}_{ab}$ represents the propagator or two-point function at tree level. Its explicit form is given below. The expansion of this expression in powers of g leads to the standard perturbative series expansion in quantum field theory. The best way to organize the different contributions is to use Feynman diagrams. These are obtained from the Feynman rules which can be read in part from (2.11). To complete the set of Feynman rules one must be more specific about the operators $\mathcal{O}(A)$. The presence of the denominator in (2.11) has a simple interpretation in terms of Feynman diagrams: one must take only those diagrams which are connected.

Before entering into the discussion on the structure of (2.11) we must first introduce the operators which will be of interest for us. We will refer to these as observables. As in any gauge theory, the observables have to be gauge invariant. In the case of a topological theory, as the one at hand, we also would like to have observables without any dependence on the metric of the manifold $\mathcal{M}$. These conditions are satisfied for the so-called Wilson loops. To introduce these objects, let us recall the notion of non-Abelian holonomy. If C is a parametrized loop in $\mathcal{M}$, then given a G connection A on $\mathcal{M}$, carrying a representation R of G, and any two points $C(s)$ and $C(t)$ on the loop, we define an element $W_C^R(s,t)$ of G in the following fashion,

$$W_C^R(s,t) = \mathrm{P}\exp\Big\{ g \int_{C(s)}^{C(t)} A \Big\}, \tag{2.12}$$

where P stands for "path ordered". This concept is analogous to the concept of time ordering in quantum field theory, and can be briefly described as follows: in the expansion of the exponential some products of connections $A(s)$ defined at different points $C(s)$ of the path will appear. The path ordering puts these factors in decreasing order of the parameter s. It is this ordering what makes the product of two such $A(s)$ to be equivalent to the above defined two-point function. For a thorough exposition of these and other common concepts in quantum field theory, see, for example, [51].

We shall denote the holonomy, or the parallel transport around the loop, by $W_C^R(s) \equiv W_C^R(s,s)$. This $W_C^R(s)$ is an element of G, and the Wilson loop is defined to be simply the trace of the holonomy of the connection 1-form A along C,

$$W_C^R = \mathrm{Tr}\Big(\mathrm{P}\exp\Big\{ g \oint_C A \Big\}\Big). \tag{2.13}$$

Notice that we have dropped the dependence on the initial point s. The trace is taken over the representation R of the algebra of G carried by the connection A. The two chief features of (2.13) are its gauge invariance and its independence on any metric whatsoever. These attributes single out the Wilson lines as the best candidates for observables in a gauge invariant topological field theory. Some generalizations of these objects are defined in [50], but will not be considered in this work. The connection of the Wilson lines with knot theory, discovered by Witten in [1], is established through the dependence of W_C^R on the loop C. This loop can be knotted in any fashion, and the vacuum expectation value of W_C^R is related to knot invariants.

The vacuum expectation values of the Wilson line are defined using (2.2) and taking (2.13) as the operator $\langle \mathcal{O} \rangle$. We will restrict ourselves in the rest of this work to the three-manifold $\mathbf{R}^3$ (so that effectively one is dealing with knot invariants on S^3). In the perturbative expansion one finds, besides convolutions as dictated by the first two Feynman rules of Fig. 1, traces of generators of the algebra G in the representation R. This fact introduces the need for an extra Feynman rule reflecting the attachment of the gauge field A to the Wilson line. This rule is the third one depicted in Fig. 1. These traces, together with part of the other two Feynman rules, generate group factors. For a given order in the perturbative expansion, the group-theoretical factors and the convolutions factorize and can be calculated independently. This fact will be of some importance in what follows, since our organization of the perturbative series is guided by the structure of the group-theoretical factors. The first Feynman rule in Fig. 1 involves the propagator $D_{ab}^{\mu\nu}(x-y)$, while the second involves the vertex $V_{abc}^{\mu\nu\rho}(x,y,z)$ or three-point function at tree level.

Actually the perturbative expansion is divergent in the sense that the convolutions of propagators and vertices are divergent integrals which need to be regularized. Moreover, the gauge invariance of (2.1) indicates that the functional integrals have to be restricted to integrals over gauge connections modulo gauge transformations. This last issue can be solved by introducing some unphysical fields (ghosts) in the action. These two problems have been thoroughly examined in the last years [26-42]. In this paper we will follow the approach taken in [28], where a Pauli-Villars regularization was introduced. We do not give the Feynman rules corresponding to ghost and Pauli-Villars fields since these fields only enter in loops and we will take the results obtained in [28] for one-loop Green functions. These results are summarized in Fig. 2. As stated in the introduction we will further assume that there exist a scheme in which the quantum corrections to the two and three-point functions account for the shift obtained in [1] of the Chern-Simons parameter k. The existence of this scheme has been proved to one loop [25,28,32,38] and to two loops [39]. This implies that we do not have to worry about Feynman diagrams containing two and three-point functions at one or higher loops.

Another important contribution inherited in the vacuum expectation value of a Wilson line is the framing factor. In our previous work [43] all diagrams contributing to this factor were identified. We will make a brief review of that result in the rest of this section. To carry this out we must introduce the following classification of propagators, which, on the other hand, will be also useful in other sections of the paper. We call "free" those propagators with both endpoints on the Wilson line, and "collapsible" those free propagators whose endpoints can get together without crossing over any point belonging to other subdiagram. For example, diagram c of Fig. 3 contains two free and one collapsible propagators.

The main result of [43] is the factorization theorem, which enables us to identify the Feynman diagrams that contribute to the framing dependence of the Wilson line. This is essential to our approach in two senses. First, the numerical knot invariants we are going to present are based on the idea that the knot invariants should not depend on the framing, which is not intrinsic to the knot, and therefore all framing dependent contributions should be isolated and discarded. Second, it explains why in other approaches to Vassiliev invariants similar types of diagrams have to be set to zero [41].

To state the factorization theorem we need to introduce some notation. We will be considering diagrams corresponding to a given order g^{2m} in the perturbative expansion of a knot, and to a given number of points running over it, namely n. We will denote by $\{i_1, i_2, \ldots, i_n\}$ a domain of integration where the order of integration is $i_1 < i_2 < ... < i_n$, being $i_1, i_2, \ldots, i_n$ the points on the knot (notice the condensed notation) where the internal lines of the diagram are attached. The integrand corresponding to that diagram will be denoted as $f(i_1, i_2, \ldots, i_n)$. Diagrams are in general composed of subdiagrams, which may be connected or non-connected. For a given diagram we can make specific choices of subdiagrams depending on the type of factorization which is intended to achieve. For example, for a diagram like c of Fig. 3 one may choose as subdiagrams the three free propagators, or one may choose a subdiagram to be the collapsible propagator and other subdiagram to be the one built by two crossed free propagators.

We will consider a set of diagrams $\mathcal{N}$ corresponding to a given order g^{2m}, to a given number of points attached to the knot, n, and to a given kind. By kind we mean all diagrams containing n_i subdiagrams of type i, $i = 1, ..., T$. By p_i we will denote the number of points which a subdiagram of type i has attached to the knot. For example, if one considers diagrams at order g^6 with $n = 6$ points attached to the knot, with three subdiagrams which are just free propagators, this set is made out of diagrams a to e of Fig. 3. However, if one considers diagrams at order g^6 with $n = 6$ with a subdiagram consisting of a free propagator and a triple vertex, this set is made out of diagrams f and g. The contribution from all diagrams in $\mathcal{N}$ can be written as the following sum:

$$\sum_{\sigma \in \Pi_n} \oint_{i_1, i_2, \ldots, i_n} f(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)}), \tag{2.14}$$

where $\sigma \in \Pi_n$, being $\Pi_n \subset P_n$ a subset of the symmetric group of n elements. Notice that Π_n reflects the different shapes of the diagrams in $\mathcal{N}$. In (2.14) the integration region has been left fixed for all the diagrams and one has introduced different integrands. One could have taken the opposite choice, namely, one could have left fixed the integrand and sum over the different domains associated to $\mathcal{N}$. The first statement regarding the factorization theorem just refers to these two possible choices. Let us define the domain resulting of permuting $\{i_1, i_2, \ldots, i_n\}$ by an element σ of the symmetric group P_n by

$$\mathcal{D}_\sigma = \{\, i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)} \,\}, \tag{2.15}$$

then the following result immediately follows.

Statement 1: *The contribution to the Wilson line of the sum of diagrams whose integrands are of the form:*

$$f(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)}), \tag{2.16}$$

where σ runs over a given subset $\Pi_n \in P_n$ with a common domain of integration is equal to the sum of the integral of $f(i_1, i_2, \ldots, i_n)$ over $\mathcal{D}_\sigma$ where $\sigma \in \Pi_n^{-1}$:

$$\oint_{i_1, i_2, \ldots, i_n} \sum_{\sigma \in \Pi_n} f(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)}) = \sum_{\sigma \in \Pi_n^{-1}} \oint_{\mathcal{D}_\sigma} f(i_1, i_2, \ldots, i_n). \tag{2.17}$$

The idea behind the factorization theorem is to organize the diagrams in $\mathcal{N}$ in such a way that one is summing over all possible permutations of domains. Summing over all domains implies that one can consider the integration over the points corresponding to each subdiagram as independent and therefore one can factorize the contribution into a product given by the integrations of each subdiagram independently. This leads to the following statement.

Statement 2: (Factorization theorem) *Let Π'_n be the set of all possible permutations of the domains of integration of diagrams containing subdiagrams of types $i = 1, ..., T$. If $\Pi_n^{-1} = \Pi'_n$, the sum of integrals over $\mathcal{D}_\sigma$, $\sigma \in \Pi_n^{-1}$, is the product of the integrals of the subdiagrams over the knot, being the domains all independent,*

$$\sum_{\sigma \in \Pi_n^{-1}} \oint_{\mathcal{D}_\sigma} f(i_1, \ldots, i_n) = \prod_{i=1}^{T} \Big(\oint_{i_1, \ldots, i_{p_i}} f_i(i_1, \ldots, i_{p_i}) \Big)^{n_i}. \tag{2.18}$$

In (2.18) n_i denotes the number of subdiagram of type i and p_i its number of points attached to the knot.

The proof of this statement is trivial since having all possible domains it is clear that one can write the integration considering subdiagram by subdiagram, the result being the product of all the partial integrations over subdiagrams.

As a consequence of the factorization theorem we can state two corollaries about the framing independence of diagrams which do not contain one-particle irreducible subdiagrams corresponding to two-point functions whose endpoints could get together. These corollaries refer to any kind of knot. Their statements are:

Framing dependent diagrams: *A diagram gives a framing dependent contribution to the perturbative expansion of the knot if and only if it contains at least one collapsible propagator. Moreover, the order of m in its contribution, the self-linking number, equals the number of collapsible propagators.*

Factorization of the framing dependence: *If all the contribution to the self-energy comes from one loop diagrams, then* $\langle W_C^R \rangle = F(C,R)\mathrm{e}^{2\pi i m h_R}$ *where* $F(C,R)$ *is framing independent but knot dependent, and the exponential is manifestly framing dependent but knot independent.*

The quantities h_R and m appearing in the framing dependence factor are, respectively, the conformal weight associated to the representation R, and the integer which labels the framing (self-linking number). The standard framing corresponds to $m = 0$. The factorization of the framing dependence was proven in [43] for $SU(N)$ in the fundamental representation but it is obvious from the proof that it generalizes for any representation of any semisimple group. We end this section recalling that a full account of these results can be found in [43]. They are the cornerstones of our approach to the finite-type invariants associated to perturbative Chern-Simons theory.

3. General structure of the perturbative expansion

In this section we will analyze the structure of the perturbative series expansion associated to Chern-Simons gauge theory with an arbitrary compact semisimple gauge group G. We will discuss the general form of this series and we will present its exact form up to order six. Let us consider the vacuum expectation value of the Wilson line (2.13) corresponding to an arbitrary knot in an arbitrary representation R of G. The contour integral in (2.13) corresponds to any path diffeomorphic to the knot. To compute the vacuum expectation values of this operator in perturbation theory we have to consider all diagrams which are not vacuum diagrams since we consider normalized vacuum expectation values, *i.e.*, the functional integration where the operator is inserted is divided by the partition function Z as in (2.2). Also, we will not consider diagrams which include collapsible propagators because they only contribute to the dependence of the vacuum expectation value of W on the framing. Finally we can omit the insertion of loops in every two and three-point subdiagram since, as stated before, their only effect is to provide the shift $k \rightarrow k - c_{\rm A}$. We stress these two last points because they greatly simplify the perturbative series. The framing and the shift are viewed as objects not intrinsic to the knot and are therefore ignored.

From the Feynman rules presented in the previous section follows that the perturbative expansion of the vacuum expectation value of the Wilson line operator (2.13) has the form,

$$\langle W_C^R \rangle = d(R) \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij} r_{ij} x^i, \tag{3.1}$$

where $x = \frac{2\pi i}{k} = ig^2/2$ is the expansion parameter, $d(R)$ is the dimension of the representation R, and $\alpha_{0,1} = r_{0,1} = 1$, $d_0 = 1$ and $d_1 = 0$. Notice that we are dispensing with the shift and the framing factor: only k appears in the denominator of x and there is no linear term in the expansion $(d_1 = 0)^{\star}$. The

⋆ At order x there is only the contribution from a diagram containing one collapsible propagator which according to the results presented in the previous section corresponds to framing.

factors α_{ij} and r_{ij} appearing at each order i incorporate all the dependence dictated from the Feynman rules apart from the dependence on the coupling constant, which is contained in x. Of these two factors, in the r_{ij} all the group-theoretical dependence is collected. These will be called group factors. The rest is contained in the α_{ij}. These last quantities, which will be called geometrical factors, have the form of integrals over the Wilson line of products of propagators, as dictated by the Feynman rules. The first index in α_{ij} denotes the order in the expansion and the second index labels the different geometrical factors which can contribute at the given order. Similarly, r_{ij} stands for the independent group structures which appear at order i which are also dictated by the Feynman rules. The object d_i in (3.1) will be called the "dimension" of the space of invariants at a given order. In our approach denotes the number of independent group structures which appear at that order. The main content of this section is the characterization of these group structures. Notice that while the geometrical factors α_{ij} are knot dependent but group and representation independent (and therefore one must keep in mind their full form $\alpha_{ij}(C)$), the group factors are group and representation dependent but knot independent ($r_{ij}(R)$).

The group factors which appear in the expansion (3.1) are group invariants made out of traces of the generators contracted with the tensors δ_{ab} and f_{abc}. These tensors are described in Appendix A. The Lie algebra (A1) satisfied by the generators and the Jacobi identity (A2) relate some of the group invariants which appear at a given order. In the expansion (3.1) the group factors r_{ij}, $j = 1, \ldots, d_i$, are the independent ones at a given order i. In general, these are obtained as follows. First one writes the group factors of all the diagrams with no collapsible propagator and no two and three-point one or higher-loop subdiagrams contributing to a given order i. Then one makes use of (A1) and (A2) so that a selected set of independent group factors is chosen.

The characterization of the independent group factors is carried out in two steps. First the independent Casimirs are constructed. Then, the independent group factors are built using these independent Casimir invariants. Casimirs invariants will be denoted by C_i^j, where the subindex denotes the order of the Casimir and the superindex labels the different Casimirs that appear at a given order. It is worth to recall here that the order i of a Casimir is such that $2i$ equals the number of generators plus the number of structure constants which appear in its expression. The independent Casimirs for a semisimple group up to order 6 turn out to be the following:

$$\begin{aligned} C_2' d(R) &= f_{apq} f_{bqp} \mathrm{Tr}(T_a T_b), \\ C_3' d(R) &= f_{apq} f_{bqr} f_{crp} \mathrm{Tr}(T_a T_b T_c), \\ C_4 d(R) &= f_{apq} f_{bqr} f_{crs} f_{dsp} \mathrm{Tr}(T_a T_b T_c T_d), \\ C_5 d(R) &= f_{apq} f_{bqr} f_{crs} f_{dst} f_{etp} \mathrm{Tr}(T_a T_b T_c T_d T_e), \\ C_6^1 d(R) &= f_{apq} f_{bqr} f_{crs} f_{dst} f_{etu} f_{rup} \mathrm{Tr}(T_a T_b T_c T_d T_e T_r), \\ C_6^2 d(R) &= f_{apq} f_{brs} f_{ctp} f_{dur} f_{eqs} f_{gtu} \mathrm{Tr}(T_a T_b T_c T_d T_e T_g). \end{aligned} \tag{3.2}$$

Several comments are in order. First, in general, if there is only one Casimir at a given order we will label it by C_i instead of C_i^1. Second, notice that the first two have been denoted by C_2' and C_3' instead of C_2 and C_3. The reason for this is that the notation C_2 and C_3 will be reserved to these two Casimirs times appropriate factors. This will simplify the expressions for the group factors. Third, the factor $d(R)$, the dimension of the representation, is introduced for convenience. Notice that for an irreducible representation there is a trace of the identity matrix on the right hand side. Fourth, it is at order 6 when two independent Casimirs appear for the first time. Notice that for a specific semisimple group these two Casimirs might not be independent. However, in general they are. What is meant by independence is that C_6^1 can not be written in terms of C_6^2 plus terms which are products of lower order Casimirs making use of (A1) and (A2). The number of independent Casimirs at order i will be denoted by c_i. We have $c_2 = c_3 = c_4 = c_5 = 1$ and $c_6 = 2$. The specific values of the independent Casimirs in (3.2) are given in Appendix A for the groups $SU(N)$ and $SO(N)$ in their fundamental representations and for $SU(2)$ in an arbitrary irreducible representation.

As announced above, the second and third order Casimirs will be redefined for later convenience. First notice that using (A2) and (A5) one finds,

$$
\begin{aligned}
C_2' d(R) &= C_A \mathrm{Tr}(T_a T_a), \\
C_3' d(R) &= -\frac{1}{2} C_A f_{abc} \mathrm{Tr}(T_a T_b T_c),
\end{aligned}
\tag{3.3}
$$

where C_A is the quadratic Casimir in the adjoint representation. On the other hand, as it will become clear below, the two traces on the right hand side of (3.3) are the quantities which more often appear in group factors. We then define,

$$
C_2 = \frac{1}{d(R)} \mathrm{Tr}(T_a T_a) = C_2'/C_A, \qquad C_3 = -\frac{1}{d(R)} f_{abc} \mathrm{Tr}(T_a T_b T_c) = 2C_3'/C_A. \tag{3.4}
$$

The diagrams associated to the independent Casimirs C_2, C_3, C_4, C_5, C_6^1 and C_6^2 are shown in Fig.4. From those diagrams one easily writes down the Casimirs using only the part of the Feynman rules concerning group-theoretical factors.

To obtain the group factors r_{ij}, $j = 1, \ldots, d_i$, to a given order one must consider all the diagrams with no collapsible propagators and no two and three-point loop-insertions, and write their group factors in terms of the independent Casimirs and products or ratios of them. We will present an algorithm which leads to the independent group factors at a given order. We will discuss first the case of simple groups and then we will generalize it for the semisimple case.

We will introduce first some notation. Let us consider an arbitrary diagram D of the perturbative expansion. We will denote by V the number of vertices, by P_{free} the number of free propagators and by P_I the number of "internal" propagators with both endpoints attached to vertices present in a given diagram. It is convenient to define he number of "effective" propagators, P, as,

$$
P = 2V - P_I + P_{\mathrm{free}}. \tag{3.5}
$$

It is also useful to introduce a grading $p(i)$ for the Casimirs of order i, C_i^k, $k = 1, \ldots, c_i$, equal to the effective number of propagators corresponding to their associated diagram (see Fig. 4).

$$p(2)=1, \qquad p(3)=2, \qquad p(i)=i \quad \text{if} \quad i \geq 4. \tag{3.6}$$

The main result leading to the characterization of group factors for the case of simple groups is the following. Given a diagram D with $P \geq 3$ effective propagators, its group-theoretical dependence takes the form,

$$r(D)=\sum_{\{S_P\}} a_{S_P} \prod_{i=2}^{P} \prod_{k=1}^{c_i} \left(C_i^k\right)^{S(i,k)}, \tag{3.7}$$

where the sum runs over all possible sets $\{S_P\}$ which we are about to define, and a_{S_P} are some rational numbers depending on the diagram D. The sets $\{S_P\}$ are all the possible collections of integers $\{S(i,k): 2 \leq i \leq P, \quad 1 \leq k \leq c_i\}$ satisfying the following conditions:

$$\begin{aligned} \sum_{i=2}^{P} \sum_{k=1}^{c_i} p(i) S(i,k) =& P, \\ S(i,k) \geq& 0 \quad \text{if} \quad i \neq 2, \\ S(2,1)+S(3,1) \geq& 0, \\ S(2,1) \geq& 2-P. \end{aligned} \tag{3.8}$$

According to (3.7), the integers $S(i,k)$ correspond to the number of times that the subdiagram associated to the Casimir C_i^k appears. The meaning of conditions (3.8) is the following. The first one simply imposes that the total number of effective propagators is correct, and the second one requires that for $i \neq 2$ the subdiagrams corresponding to the Casimirs C_i^k appear a positive number of times. The possible negative values of the $S(i,k)$ for $i=2$ are due to the presence of C_A factors, as explained below. Finally, the last two inequalities are the constraints on these values. Notice that we have used the fact that $c_2=c_3=1$. The formula (3.7) is valid for $P \geq 3$. The group structures corresponding to $P=2$ must be obtained independently. However, as discussed below, these are very simple. Finally, we must mention that (3.7) provides all the group structures, including the ones contributing to framing. These will be identified and discarded thereafter.

The proof of (3.7) goes by induction in P. Assume that we are given a diagram D_p with $P = p$ as the one depicted in Fig. 5 and that its group factor is $r(D_p)$ according to (3.7).

Now one more free propagator is introduced, and the key idea is to move one of its endpoints next to the other by means of the commutation relations. This produces a series of diagrams with $p+1$ effective propagators. The last element of this series would include a diagram similar to Fig. 5 with a collapsible propagator included, as the one depicted in Fig. 6. Its group factor will be $C_2\, r(D_p)$.

According to our general framework, this is a framing dependent contribution which should be set to zero, but we are interested only in group factors at this moment. The remaining diagrams can be represented as the one in Fig.7.

The next step is based on the observation of the fact that all diagrams like Fig. 7 have a vertex with two points attached to the Wilson line. Again the procedure employs the commutation relations in order to make these points closer until we finish in a diagram with a "fishtail" like the one on the left of Fig. 8. By fishtail we mean a configuration of internal lines in a diagram such the two propagators attached contiguously to the Wilson line are joined to the same vertex. The other diagrams generated are similar to the one on the right in the same figure.

As explained in Appendix B fishtails amount to a factor $C_A/2 = -C_3/C_2$ and therefore the group factor corresponding to the first diagram in Fig. 8 can be written as $-(C_3/C_2)\, r(D_p)$. The general procedure reproduces this scheme. Take one of the vertices generated in the previous steps such that it has two points connected to the Wilson line and move them until one gets a fishtail. Then repeat the same routine with the diagrams arisen due to those movings. The procedure finishes when one is left with diagrams such that all vertices have at most one point on the Wilson line. This describes a diagram with closed loops which can be connected or disconnected, as is schematically shown in Fig. 9. In the first case it can be written as a combination of the Casimirs in C^i_{p+1} and in the second as a combination of products of lower Casimirs with a total number of effective propagators equal to $p+1$. These diagrams will be referred to as "Casimir-like".

The result is that the group factors needed for $r(D_{p+1})$ are constructed by multiplying those of $r(D_p)$ by C_2 or C_3/C_2, by considering products of lower Casimirs with a total number of effective propagators equal to $p+1$, and by including the new Casimirs C^k_{p+1}. In our notation this means that P can be increased in one unit only by some of the following procedures: adding 1 to $S(2,1)$; adding 1 to $S(3,1)$ and -1 to $S(2,1)$; giving values to the $S(i,k)$ corresponding to lower Casimirs in such a way that the number of effective propagators equals $p+1$; finally, setting $S(p+1,k)=1$ (only one k at a time) for the new Casimirs. This is verified by (3.7) for $P=p+1$ if the numbers $S(i,k)$ satisfy (3.8).

To end the proof of (3.7) we must present the group structures for the case $P=3$. Let us discuss first, as promised, the ones for $P=2$. This offers no difficulty because this case is nearly trivial. The diagrams contributing for $P=2$ are the ones in Fig. 10 plus the diagram containing two collapsible propagators, which has not been pictured. The calculation of the group factors corresponding to these diagrams is rather straightforward and they turn out to be $(C_2)^2$ and C_3. For $P=3$ one has group factors from the diagrams presented in Fig. 3. Although a larger set than in the $P=2$ case, their group factors are also computed very easily. The independent ones are:

$$
\begin{aligned}
P=3: \qquad & (3,0) \quad \longrightarrow \Big(\sum C_2\Big)^3 \\
& (1,1) \quad \longrightarrow \Big(\sum C_2\Big)\sum C_3 \\
& (-1,2) \quad \longrightarrow \sum\Big[\big(C_2\big)^{-1}\big(C_3\big)^2\Big] \quad *
\end{aligned}
$$

where we have used the symbol $\sum C^k_i$ as a shorthand for $\sum_{l=1}^n C_i^{k\,(l)}$. A similar condensed notation will be adopted in the examples presented below. An asterisk signals the group factors which contribute to the framing independent part. It follows from the structure of the proof that these will be all group factors which do not contain positive powers of $\sum C_2$. The numerical sequence written on the left of the group factors correspond to the set of integers $S(i,k)$. Clearly, they satisfy (3.8) and the proof of (3.7) is completed.

The generalization to the semisimple case can be treated as follows. It may happen that different diagrams lead to the same group structure in the simple case. This is explained in Appendix B. In (3.7) the products of Casimirs can sometimes be separated in subproducts in such a way that each subproduct corresponds to a possible subdiagram. These decompositions would yield all diagrams which in the simple case have the same group factor. If the algebra is now $A = \oplus_{l=1}^{n} A_l$ the Casimirs are $C_i^k = \sum_{l=1}^{n} C_i^{k\,(l)}$, and we can put the sum over l in front of each subproduct for each of the decompositions. This would enlarge the number of independent group structures.

Each of the subproducts appearing in a group factor corresponding to a given number of effective propagators P verifies again (3.8) but with a smaller P, namely P^j. If the number of subproducts is s it has to be satisfied that,

$$\sum_{j=1}^{s} P^j = P. \tag{3.9}$$

It should be noted that if the number c_i in (3.8) and (3.7) were known, our construction would provide a systematic method to compute the dimension d_i. The numbers c_i can probably be calculated by purely group-theoretical methods. We leave this problem open for future work.

To clarify how (3.7) and the algorithm implicit in the proof works, some examples are in order. We will present explicit calculations of the sequences $\{S_P\}$ for $P \leq 6$, including their corresponding factors r_{ij}, which can be read from (3.7). As in the case $P = 3$, we will write integer sequences following the pattern $(S(2,1), S(3,1), S(4,1), S(5,1), [S(6,1), S(6,2)], \ldots)$, *i.e.*, in growing order in i and for a given i in growing order in k. The $S(i,k)$ corresponding to a given i will be gathered inside square brackets. As used above, an asterisk will indicate that the corresponding group structure contributes to the framing independent part.

The case $P = 4$ is the first one which follows the algorithm. The sequences $S(i,k)$ are the only possible ones verifying (3.8), and can be constructed starting from the data for $P = 3$ either by adding 1 to $S(2,1)$ or by adding -1 to $S(2,1)$ and 1 to $S(3,1)$. In addition, the third possibility pointed out in the proof of (3.7) has to be taken into account because there is a Casimir corresponding to four effective propagators, C_4, and therefore the sequence $(0,0,1)$ must be included:

$$
\begin{aligned}
P = 4: \qquad & (4,0,0) \longrightarrow \big(\textstyle\sum C_2\big)^4 \\
& (2,1,0) \longrightarrow \big(\textstyle\sum C_2\big)^2 \textstyle\sum C_3 \\
& (0,2,0) \longrightarrow \begin{cases} \big(\sum C_3\big)^2 \quad * \\ \big(\sum C_2\big) \sum \big[\big(C_2\big)^{-1}\big(C_3\big)^2\big] \end{cases} \\
& (-2,3,0) \longrightarrow \textstyle\sum \big[\big(C_2\big)^{-2}\big(C_3\big)^3\big] \quad * \\
& (0,0,1) \longrightarrow \textstyle\sum C_4 \quad *
\end{aligned}
$$

Notice that the sequence $(0,2,0)$ contains more than one group structure. This happen after using the generalization of the algorithm for the semisimple case which has been described. If the algebra were simple these multiple structures would reduce to only one, as can be seen after suppressing the sums over the simple components of the algebra. To cope with these multiplicities we follow the procedure outlined in the generalization of the algorithm to semisimple algebras. First one writes the group structures corresponding to the simple case. These are: $\big(C_2\big)^4$, $\big(C_2\big)^2 C_3$, $\big(C_3\big)^2$, $\big(C_2\big)^{-2}\big(C_3\big)^3$, C_4. Then one considers all possible partitions of the previous factors in such a way that the subfactors correspond to subdiagrams. These subfactors correspond to a smaller number of effective propagators, as explained in (3.9). Once this is done, a sum over the simple components of the algebra can be put in front of each subfactor. In the case at hand this can only be done in $\big(C_3\big)^2$ which can also be written as $\big[C_2\big]\big[\big(C_3\big)^2\big(C_2\big)^{-1}\big]$. Each of these two partitions correspond to admissible decompositions in subdiagrams. The first partition can be represented by two separated subdiagrams which are three vertices, and the second by one collapsible propagator besides a subdiagram like the diagram h of Fig. 3. Although these two different decompositions lead to the same group factor in the simple case, the semisimple case distinguishes them. Putting the sum over simple components of the algebra in front of each subfactor we get the two different group structures corresponding to the sequence $(0, 2, 0)$. Similar reasonings are followed in the next cases.

The cases $P = 5$ and $P = 6$ are obtained following the same procedure. We present here the corresponding results:

$$
\begin{aligned}
P = 5: \qquad & (5,0,0,0) \longrightarrow \left(\sum C_2\right)^5 \\
& (3,1,0,0) \longrightarrow \left(\sum C_2\right)^3 \sum C_3 \\
& (1,2,0,0) \longrightarrow \begin{cases} \left(\sum C_2\right)\left(\sum C_3\right)^2 \\ \left(\sum C_2\right)^2 \left(\sum \left[(C_2)^{-1}(C_3)^2\right]\right. \end{cases} \\
& (1,0,1,0) \longrightarrow \left(\sum C_2\right) \sum C_4 \\
& (-1,1,1,0) \longrightarrow \sum \left[(C_2)^{-1} C_3 C_4\right] \quad * \\
& (-1,3,0,0) \longrightarrow \begin{cases} \sum \left[(C_2)^{-1}(C_3)^2\right] \sum C_3 \quad * \\ \left(\sum C_2\right) \sum \left[(C_2)^{-2}(C_3)^3\right] \end{cases} \\
& (-3,4,0,0) \longrightarrow \sum \left[(C_2)^{-3}(C_3)^4\right] \quad * \\
& (0,0,0,1) \longrightarrow \sum C_5 \quad *
\end{aligned}
$$

$$
\begin{aligned}
P=6: \quad & (6,0,0,0,[0,0]) && \longrightarrow \big(\sum C_2\big)^6 \\
& (4,1,0,0,[0,0]) && \longrightarrow \big(\sum C_2\big)^4 \sum C_3 \\
& (2,2,0,0,[0,0]) && \longrightarrow \begin{cases} \big(\sum C_2\big)^2\big(\sum C_3\big)^2 \\ \big(\sum C_2\big)^3 \sum \big[(C_2)^{-1}(C_3)^2\big] \end{cases} \\
& (2,0,1,0,[0,0]) && \longrightarrow \big(\sum C_2\big)^2 \sum C_4 \\
& (1,0,0,1,[0,0]) && \longrightarrow \big(\sum C_2\big) \sum C_5 \\
& (0,3,0,0,[0,0]) && \longrightarrow \begin{cases} \big(\sum C_3\big)^3 \quad * \\ \big(\sum C_2\big)\big(\sum C_3\big) \sum \big[(C_2)^{-1}(C_3)^2\big] \\ \big(\sum C_2\big)^2 \sum \big[(C_2)^{-2}(C_3)^3\big] \end{cases} \\
& (0,1,1,0,[0,0]) && \longrightarrow \begin{cases} \big(\sum C_3\big) \sum C_4 \quad * \\ \big(\sum C_2\big) \sum \big[(C_2)^{-1} C_3 C_4\big] \end{cases} \\
& (-1,1,0,1,[0,0]) && \longrightarrow \sum \big[(C_2)^{-1} C_3 C_5\big] \quad * \\
& (-2,2,1,0,[0,0]) && \longrightarrow \sum \big[(C_2)^{-2}(C_3)^2 C_4\big] \quad * \\
& (-2,4,0,0,[0,0]) && \longrightarrow \begin{cases} \big(\sum C_2\big) \sum \big[(C_2)^{-3}(C_3)^4\big] \\ \big(\sum C_3\big) \sum \big[(C_2)^{-2}(C_3)^3\big] \quad * \\ \big(\sum \big[(C_2)^{-1}(C_3)^2\big]\big)^2 \quad * \end{cases} \\
& (-4,5,0,0,[0,0]) && \longrightarrow \sum \big[(C_2)^{-4}(C_3)^5\big] \quad * \\
& (0,0,0,0,[1,0]) && \longrightarrow \sum C_6^1 \quad * \\
& (0,0,0,0,[0,1]) && \longrightarrow \sum C_6^2 \quad *
\end{aligned}
$$

With the help of the results presented for $P = 1, \ldots, 6$, we are in the position to write (3.1) explicitly up to order six:

$$
\begin{aligned}
\langle W_C^R \rangle = d(R)\Big[& 1 + \alpha_{2,1} r_{2,1} x^2 + \alpha_{3,1} r_{3,1} x^3 \\
& + \big(\alpha_{4,1} r_{2,1}^2 + \alpha_{4,2} r_{4,2} + \alpha_{4,3} r_{4,3}\big) x^4 \\
& + \big(\alpha_{5,1} r_{2,1} r_{3,1} + \alpha_{5,2} r_{5,2} + \alpha_{5,3} r_{5,3} + \alpha_{5,4} r_{5,4}\big) x^5 \\
& + \big(\alpha_{6,1} r_{2,1}^3 + \alpha_{6,2} r_{3,1}^2 + \alpha_{6,3} r_{2,1} r_{4,2} + \alpha_{6,4} r_{2,1} r_{4,3} + \alpha_{6,5} r_{6,5} \\
& + \alpha_{6,6} r_{6,6} + \alpha_{6,7} r_{6,7} + \alpha_{6,8} r_{6,8} + \alpha_{6,9} r_{6,9}\big) x^6 + O(x^7)\Big].
\end{aligned}
\tag{3.10}
$$

The group factors r_{ij} can be read from the listed results above. These can be classified in two types, the ones which are not products of lower order group factors,

$$
\begin{aligned}
r_{2,1} &= \sum_{k=1}^{n} C_3^{(k)}, & r_{5,4} &= \sum_{k=1}^{n} C_5^{(k)}, \\
r_{3,1} &= \sum_{k=1}^{n} \big(C_3^{(k)}\big)^2 \big(C_2^{(k)}\big)^{-1}, & r_{6,5} &= \sum_{k=1}^{n} \big(C_3^{(k)}\big)^5 \big(C_2^{(k)}\big)^{-4}, \\
r_{4,2} &= \sum_{k=1}^{n} \big(C_3^{(k)}\big)^3 \big(C_2^{(k)}\big)^{-2}, & r_{6,6} &= \sum_{k=1}^{n} C_4^{(k)} \big(C_3^{(k)}\big)^2 \big(C_2^{(k)}\big)^{-2}, \\
r_{4,3} &= \sum_{k=1}^{n} C_4^{(k)}, & r_{6,7} &= \sum_{k=1}^{n} C_5^{(k)} C_3^{(k)} \big(C_2^{(k)}\big)^{-1}, \\
r_{5,2} &= \sum_{k=1}^{n} \big(C_3^{(k)}\big)^4 \big(C_2^{(k)}\big)^{-3}, & r_{6,8} &= \sum_{k=1}^{n} C_6^{1\,(k)}, \\
r_{5,3} &= \sum_{k=1}^{n} C_4^{(k)} C_3^{(k)} \big(C_2^{(k)}\big)^{-1}, & r_{6,9} &= \sum_{k=1}^{n} C_6^{2\,(k)},
\end{aligned}
\tag{3.11}
$$

and the ones which are products of lower order ones,

$$
\begin{aligned}
r_{4,1} &= r_{2,1}^2, & r_{6,2} &= r_{3,1}^2, \\
r_{5,1} &= r_{2,1} r_{3,1}, & r_{6,3} &= r_{2,1} r_{4,2}, \\
r_{6,1} &= r_{2,1}^3, & r_{6,4} &= r_{2,1} r_{4,3}.
\end{aligned}
\tag{3.12}
$$

In Fig. 11 a representative diagram for each of these group factors has been pictured. These are easily obtained from (3.11) and (3.12) after taking into account the diagrams corresponding to the independent Casimirs drawn in Fig. 4.

From the results (3.11) and (3.12) one can read off the values of the dimensions d_i for $i = 1$ to 6:

$$
d \quad = \quad 0, \quad 1, \quad 1, \quad 3, \quad 4, \quad 9. \tag{3.13}
$$

The value $d_1 = 0$ was implicit in (3.10) and its origin resides on the absence of the term linear in x in that equation, which is due to the withdrawal of the framing.

The expansion (3.1) verifies a basic property closely related to the factorization theorem. If we denote by $\langle W_C^{R_k}\rangle$ and $\langle W_C^{R}\rangle$ the vacuum expectation values of Wilson lines based on the algebras A_k and $A = \oplus_{k=1}^{n} A_k$, being the representation R a direct product of representations R_k, it turns out that,

$$\langle W_C^{R}\rangle = \prod_{k=1}^{n} \langle W_C^{R_k}\rangle. \tag{3.14}$$

This follows directly from the factorization of both the partition function and the Wilson line operator. Comparing both sides of (3.14) some relations among the geometric coefficients in their corresponding expansions (3.1) appear, which can be also derived after knowing their explicit form by means of the factorization theorem:

$$\begin{aligned} \alpha_{4,1} &= \frac{1}{2}\alpha_{2,1}^2, & \alpha_{6,2} &= \frac{1}{2}\alpha_{3,1}^2, \\ \alpha_{5,1} &= \alpha_{2,1}\alpha_{3,1}, & \alpha_{6,3} &= \alpha_{2,1}\alpha_{4,2}, \\ \alpha_{6,1} &= \frac{1}{6}\alpha_{2,1}^3, & \alpha_{6,4} &= \alpha_{2,1}\alpha_{4,3}. \end{aligned} \tag{3.15}$$

These relationships hold also if the Wilson line corresponding to the knot under consideration is normalized by the Wilson line of the unknot, although the specific values of the α_{ij} change. This will be proven in the next section. These properties are explicitly checked in several examples in the next section.

4. The numerical knot invariants

So far we have been concerned mainly with the group-theoretical aspects of the perturbative series. Now the numerical coefficients α_{ij} are analyzed. As stated in the introduction they correspond to a series of line and three-dimensional integrals of certain integrand dictated by the Feynman rules of Fig. 1. More explicitly, α_{ij} is the sum of the geometric terms of all diagrams whose group factor contains r_{ij}. The easiest non-trivial example is $\alpha_{2,1}$, which receives contributions from the diagrams presented in Fig. 10. These diagrams have the following group factors:

$$\begin{aligned}\mathrm{Tr}(T_aT_bT_aT_b) =& \big(\sum C_2\big)^2 d(R) + r_{2,1}d(R),\\ f_{abc}\mathrm{Tr}(T_aT_bT_c) =& - r_{2,1}d(R),\end{aligned} \tag{4.1}$$

and therefore, after extracting x^2, we can write down a concrete expression for $\alpha_{2,1}$,

$$\begin{aligned}\alpha_{2,1}(C) =& \frac{1}{4\pi^2}\oint_C dx_\mu \int^x dy_\nu \int^y dz_\rho \int^z dw_\tau \epsilon^{\mu\sigma_1\rho}\epsilon^{\nu\sigma_2\tau}\frac{(x-z)_{\sigma_1}}{|x-z|^3}\frac{(y-w)_{\sigma_2}}{|y-w|^3}\\ &-\frac{1}{16\pi^3}\oint_C dx_\mu \int^x dy_\nu \int^y dz_\rho \int_{\mathbf{R}^3} d^3\omega\Big(\epsilon^{\mu\rho_1\sigma_1}\epsilon^{\nu\rho_2\sigma_2}\epsilon^{\rho\rho_3\sigma_3}\epsilon_{\sigma_1\sigma_2\sigma_3}\frac{(x-w)_{\rho_1}}{|x-w|^3}\\ &\qquad\frac{(y-w)_{\rho_2}}{|y-w|^3}\frac{(z-w)_{\rho_3}}{|z-w|^3}\Big).\end{aligned} \tag{4.2}$$

This quantity was first studied in the context of perturbative Chern-Simons gauge theory in [27]. It turns out that it is related to the second coefficient in Conway's version of the Alexander polynomial, and to the Arf and Casson invariants [52] (see also [41]). Its value for the unknot, the right-handed trefoil and the figure-eight knots on the three-manifold S^3 are [27]: $-1/6$, $23/6$ and $-25/6$ respectively. In [27] it was shown explicitly that $\alpha_{2,1}$ is framing independent. This is guaranteed in our approach since all framing dependence has been removed from (3.1) after the identification of diagrams which contribute to framing done in sect. 2. Actually, this argument extends to any α_{ij}, *i.e.*, all these invariants are framing independent. This invariant is in some sense the paradigm of the finite-type invariants arisen from perturbative Chern-Simons theory. The next invariant is $\alpha_{3,1}$. The diagrams needed are d, e, f and h of Fig. 3 whose group factors are

$$\begin{aligned}
\mathrm{Tr}(T_aT_bT_aT_cT_bT_c) =&\big(\sum C_2\big)^3 d(R) + 2\big(\sum C_2\big) r_{2,1} d(R) + r_{3,1} d(R),\\
\mathrm{Tr}(T_aT_bT_cT_aT_bT_c) =&\big(\sum C_2\big)^3 d(R) + 3\big(\sum C_2\big) r_{2,1} d(R) + 2r_{3,1} d(R),\\
f_{abc}\mathrm{Tr}(T_aT_bT_dT_cT_d) =&\big(\sum C_2\big) r_{2,1} d(R) + r_{3,1} d(R),\\
f_{abr}f_{rcd}\mathrm{Tr}(T_aT_bT_cT_d) =& r_{3,1} d(R).
\end{aligned} \tag{4.3}$$

Taking into account that there are 3 diagrams of type d, 1 of type e, 5 of type f and 2 of type h in the perturbative expansion, the integral representation of $\alpha_{3,1}$ reads

$$\begin{aligned}
\alpha_{3,1}(C) =& \frac{3}{8\pi^3}\oint_C dx_\mu \int^x dy_\nu \int^y dt_\rho \int^t dz_\tau \int^z dv_\eta \int^v dw_\zeta \epsilon^{\mu\sigma_1\tau}\epsilon^{\nu\sigma_2\zeta}\epsilon^{\rho\sigma_3\eta}\\
& \frac{(x-z)_{\sigma_1}}{|x-z|^3}\frac{(y-w)_{\sigma_2}}{|y-w|^3}\frac{(t-v)_{\sigma_3}}{|t-v|^3}\\
&+\frac{1}{4\pi^3}\oint_C dx_\mu \int^x dy_\nu \int^y dt_\rho \int^t dz_\tau \int^z dv_\eta \int^v dw_\zeta \epsilon^{\mu\sigma_1\tau}\epsilon^{\nu\sigma_2\eta}\epsilon^{\rho\sigma_3\zeta}\\
& \frac{(x-z)_{\sigma_1}}{|x-z|^3}\frac{(y-v)_{\sigma_2}}{|y-v|^3}\frac{(t-w)_{\sigma_3}}{|t-w|^3}\\
&+\frac{5}{32\pi^4}\oint_C dx_\mu \int^x dy_\nu \int^y dt_\rho \int^t dz_\tau \int^z dv_\eta \int_{\mathbf{R}^3} d^3\omega \epsilon^{\nu\sigma\eta}\epsilon_{\alpha\beta\gamma}\epsilon^{\mu\sigma_1\alpha}\epsilon^{\rho\sigma_2\beta}\epsilon^{\tau\sigma_3\gamma}\\
& \frac{(y-v)_\sigma}{|y-w|^3}\frac{(x-\omega)_{\sigma_1}}{|x-\omega|^3}\frac{(t-\omega)_{\sigma_2}}{|t-\omega|^3}\frac{(z-\omega)_{\sigma_3}}{|z-\omega|^3}\\
&+\frac{1}{64\pi^5}\oint_C dx_\mu \int^x dy_\nu \int^y dt_\rho \int^t dz_\tau \int_{\mathbf{R}^3} d^3\omega_1 \int_{\mathbf{R}^3} d^3\omega_2 \epsilon_{\alpha\beta\gamma}\epsilon_{\eta\xi\zeta}\epsilon^{\mu\sigma_1\alpha}\\
& \epsilon^{\nu\sigma_2\beta}\epsilon^{\gamma\sigma_3\zeta}\epsilon^{\rho\sigma_4\eta}\epsilon^{\tau\sigma_5\xi}\frac{(x-\omega_1)_{\sigma_1}}{|x-\omega_1|^3}\frac{(y-\omega_1)_{\sigma_2}}{|y-\omega_1|^3}\frac{(\omega_1-\omega_2)_{\sigma_3}}{|\omega_1-\omega_2|^3}\frac{(t-\omega_2)_{\sigma_4}}{|t-\omega_2|^3}\frac{(z-\omega_2)_{\sigma_5}}{|z-\omega_2|^3}
\end{aligned} \tag{4.4}$$

An important property of the geometrical factors α_{ij} is their behavior under changes of orientation in the manifold. Notice that while $\alpha_{2,1}$ possesses a product of an even number of three-dimensional totally antisymmetric tensors in all its terms, $\alpha_{3,1}$ has a product of an odd number. Thus, under a change of orientation $\alpha_{2,1}$ is even and $\alpha_{3,1}$ is odd. From the Feynman rules of the theory follows that, in general, for even i the factors α_{ij} are even under a change of orientation while for odd i those factors are odd. This implies that a knot K and its mirror image $\tilde{K}$ have geometrical factors such that

$$\begin{aligned} \alpha_{ij}(K) &= \alpha_{ij}(\tilde{K}), \quad \text{if} \quad i \quad \text{is even}, \\ \alpha_{ij}(K) &= -\alpha_{ij}(\tilde{K}), \quad \text{if} \quad i \quad \text{is odd}. \end{aligned} \tag{4.5}$$

In particular, for amphicheiral knots ($K \sim \tilde{K}$), $\alpha_{ij}(K) = 0$ for i odd. Looking back at the expansion (3.1) one observes that these results are in agreement with the fact that for quantum group knot invariants, their value for knots related by a change of orientation in the manifold are the same once the replacement $q \to q^{-1}$ is performed. Since $q = \mathrm{e}^x$, this is equivalent to carry out the change $x \to -x$, which, using (3.1) and the fact that the r_{ij} are knot independent implies (4.5).

It is possible to continue the procedure described above and give the expressions which correspond to higher coefficients α_{ij}. One simply has to draw all diagrams corresponding to the given order, compute their group factors in the way explained above, gather the framing independent contributions and display an integral after using the Feynman rules. Nevertheless the resulting expressions are somewhat unwieldy and not too illuminating. We will instead study the properties of these knot invariants.

The knot invariants α_{ij} can be written in many ways because their defining expansion (3.1) is subject to two different types of normalizations. On the one hand, the vacuum expectation value $\langle W_C^R \rangle$ could be normalized differently. For example, the choice made in (3.1) is such that it does not have value one for the unknot. Dividing $\langle W_C^R \rangle$ by the corresponding quantity for the unknot will shift the values of the α_{ij}. On the other hand, the group factors depend on the group theoretical conventions, in particular the normalization of the generators of the semisimple group. Since $\langle W_C^R \rangle$ is independent on how those generators are normalized, the α_{ij} must be different for different normalizations. In more explicit terms, the integral expressions obtained for $\alpha_{2,1}$ and for $\alpha_{3,1}$ in (4.2) and (4.4) would contain different global factors. To make the knot invariants α_{ij} universal we will first redefine them dividing by the unknot. This will fix the additive arbitrariness of the α_{ij} and will impose the property that all these invariants vanish for the unknot. Second, we will fix the multiplicative arbitrariness of the invariants α_{ij} by taking the simplest non-trivial knot, the trefoil, and fixing the values of the invariants to some selected integers. This can be done if all the invariants do not vanish for the trefoil. This holds up to order six and we will assume that it holds in general. As we will see in the next section the choice made supports the conjecture that it is possible to find a normalization were all the invariant quantities are integers. This is a highly non-trivial feature looking at their integral representations as the ones in (4.2) and (4.4). In the next section we will present all these facts explicitly up to order six for all prime knots up to six crossings.

Let us denote the unknot by U and let us consider its expansion (3.1),

$$\langle W_U^R \rangle = d(R) \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij}(U) r_{ij}(R) x^i. \tag{4.6}$$

Let us now consider un arbitrary knot K. We define the new knot invariants $\tilde{\alpha}_{ij}(K)$ normalizing by the expression for the unknot:

$$\frac{\langle W_K^R \rangle}{\langle W_U^R \rangle} = \frac{\sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij}(K) r_{ij}(R) x^i}{\sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij}(U) r_{ij}(R) x^i} = \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \tilde{\alpha}_{ij}(K) r_{ij}(R) x^i. \tag{4.7}$$

Similarly to the case of (3.1), one has $\tilde{\alpha}_{0,1} = 1$. Notice that for the unknot, U, one has $\tilde{\alpha}_{ij}(U) = 0$, $\forall\, i, j$, such that $i \neq 0$. From the values given above for $\alpha_{2,1}$ for the unknot, the right-handed trefoil and the figure-eight knot ($-1/6$, $23/6$ and $-25/6$

respectively), one easily obtain the value of $\tilde{\alpha}_{2,1}$ for the right-handed trefoil and the figure-eight knot: $23/6 + 1/6 = 4$ and $-25/6 + 1/6 = -4$, respectively. It is clear from (4.7) and the fact that the unknot is amphicheiral that the properties (4.5) are also satisfied by the $\tilde{\alpha}_{ij}$.

The new quantities $\tilde{\alpha}_{ij}(K)$ also satisfy relations as the ones in (3.15). To prove this notice that relation (3.14) holds for any knot, in particular for the unknot. This implies,

$$\frac{\langle W_C^R \rangle}{\langle W_U^R \rangle} = \frac{\prod_{k=1}^n \langle W_C^{R_k} \rangle}{\prod_{k=1}^n \langle W_U^{R_k} \rangle} = \prod_{k=1}^n \frac{\langle W_C^{R_k} \rangle}{\langle W_U^{R_k} \rangle}, \tag{4.8}$$

which, similarly to the case (3.15) leads to,

$$\begin{aligned} \tilde{\alpha}_{4,1} &= \frac{1}{2}\tilde{\alpha}_{2,1}^2, & \tilde{\alpha}_{6,2} &= \frac{1}{2}\tilde{\alpha}_{3,1}^2, \\ \tilde{\alpha}_{5,1} &= \tilde{\alpha}_{2,1}\tilde{\alpha}_{3,1}, & \tilde{\alpha}_{6,3} &= \tilde{\alpha}_{2,1}\tilde{\alpha}_{4,2}, \\ \tilde{\alpha}_{6,1} &= \frac{1}{6}\tilde{\alpha}_{2,1}^3, & \tilde{\alpha}_{6,4} &= \tilde{\alpha}_{2,1}\tilde{\alpha}_{4,3}. \end{aligned} \tag{4.9}$$

The knot invariants $\tilde{\alpha}_{ij}$, besides being topological invariants and framing independent, are also knot invariants of finite type in the sense of Vassiliev. This follows from the results of [48], where the authors showed that the i^{th} coefficient of the expansion in x of the HOMFLY and Kauffman polynomials of an arbitrary knot K, $H_{N,q}(K)$ and $R_{N,q}(K)$, after taking $q = \mathrm{e}^x$, is a Vassiliev invariant of order i. The theorem was extended for an arbitrary quantum group invariant in [49]. It is also presented in [41] and, in full generality, in [53]. We simply extend this result to the different coefficients $\tilde{\alpha}_{ij}$ which contribute to the i^{th} order in the expansion. The idea is that at a given order the different structures r_{ij} are independent and therefore all the $\tilde{\alpha}_{ij}$ are independent Vassiliev invariants of order i.

To be self-contained, we review very briefly the axiomatic approach to Vassiliev invariants proposed in [48] and [53], where a thorough treatment of the subject can be found. A j-singular knot is a knot which has j transversal self-intersections. This object is denoted by K^j. The self-intersection can be made an undercrossing or an overcrossing, which are called the resolutions of the self-intersection. Given a knot invariant $V(K)$ it can be extended to be an invariant of j-singular knots by means of the prescription presented in Fig. 12.

The formula presented in Fig. 12 is the first axiom of Birman and Lin. If we denote by K^j_+ a j-singular knot with an overcrossing at a given point and by K^j_- the same with an undercrossing instead of the overcrossing, this axiom can be written as a crossing-change formula,

$$V(K^j) = V(K^{j-1}_+) - V(K^{j-1}_-). \tag{4.10}$$

The second axiom states that the Vassiliev invariants vanish on j-singular knots for j high enough. In other words:

$$\exists i \in Z^+ \qquad \text{such that} \qquad V(K^j) = 0 \qquad \text{if} \qquad j > i. \tag{4.11}$$

The smallest such i is called the order (or type) of V. A Vassiliev invariant of order i will be denoted by V_i. Besides these two axioms, some initial data are needed. Let us denote by U the unknot. Then one requires that all Vassiliev invariants of U vanish,

$$V_i(U) = 0, \qquad \forall i \in Z^+. \tag{4.12}$$

A singular point p is called nugatory if its two resolutions define the same knot. Let us denote by K^j_p a singular knot which includes a nugatory point p. If we are to obtain knot invariants, the following axiom has to be satisfied:

$$V_i(K^j_p) = 0, \qquad \text{if } p \text{ is nugatory}. \tag{4.13}$$

A further set of initial data is needed to begin with the calculation of the invariants. This corresponds to a set of given $V_i(K^j)$ for some selected singular knots, presented in the form of a table, called the actuality table. Of course these numbers are not arbitrary and have to satisfy some rules in order to yield consistent values for the numerical invariants. These consistency conditions are a system of linear equations, where the unknowns are the numbers present in the actuality table. Birman and Lin [48,53] proved that the expansion of any quantum group invariant associated to a knot K yields consistent values for these numerical knot invariants, and therefore are Vassiliev invariants. We will use this result to prove that the knot invariants $\tilde{\alpha}_{ij}$ in (4.7) are Vassiliev invariants.

Let us consider a knot K. According to Birman and Lin [48,49], for any semisimple group, one can assert that the contribution at each order in x in the expansion (3.1),

$$V_i(K) = \sum_{k=1}^{d_i} \tilde{\alpha}_{ik}(K) r_{ik}, \tag{4.14}$$

is a Vassiliev invariant of order i. This means that these V_i satisfy the axioms given above. In other words, if one defines invariants for j-singular knots from (4.14) using (4.10), one finds that (4.11), (4.12) and (4.13) hold, and that a consistent actuality table is obtained. Following the same mechanism we define the $\tilde{\alpha}_{ik}(K)$ for j-singular knots using (4.14) and (4.10). Indeed, in writing (4.10), one finds,

$$\sum_{k=1}^{d_i} \tilde{\alpha}_{ik}(K^j) r_{ik} = \sum_{k=1}^{d_i} \tilde{\alpha}_{ik}(K_+^{j-1}) r_{ik} - \sum_{k=1}^{d_i} \tilde{\alpha}_{ik}(K_-^{j-1}) r_{ik}, \tag{4.15}$$

and then, from the independence of the group factors r_{ik} follows that,

$$\tilde{\alpha}_{ik}(K^j) = \tilde{\alpha}_{ik}(K_+^{j-1}) - \tilde{\alpha}_{ik}(K_-^{j-1}). \tag{4.16}$$

To prove that the quantities $\tilde{\alpha}_{ik}(K^j)$ defined in this way satisfy (4.11), (4.12) and (4.13) we must take into account that, again, making use of the theorem by Birman and Lin, $V_i(K^j) = \sum_{k=1}^{d_i} \tilde{\alpha}_{ik}(K^j) r_{ik}$ do satisfy these axioms. Writting out the corresponding expressions in terms of this sum and making use of the independence of the group factors follows that (4.11), (4.12) and (4.13) are also verified by the quantities $\tilde{\alpha}_{ik}(K^j)$. Similarly, one concludes that given a type i there is a consistent actuality table for each k, *i.e.*, the $\tilde{\alpha}_{ik}$ generate an actuality table once their value for j-singular knots is defined through (4.16). Therefore, the geometrical factors associated to knots in (4.7) and their extension to j-singular knots done in (4.16) are invariants of finite type or Vassiliev invariants. Notice that these invariants generate d_i actuality tables for each i. The actuality table that one would generate following Birman and Lin for a given quantum group invariant would be a special linear combination of these d_i actuality tables.

Vassiliev invariants form an algebra, not just a sequence of vector spaces. Products of Vassiliev invariants of types i and j lead to Vassiliev invariants of type ij. This structure is also manifest in our knot invariants due to relations (4.9). These follow from the factorization property of Wilson lines for semisimple groups (3.14). Invariants can in this way be classified in two types: simple invariants as the ones which are not product of lower type invariants, and compound invariants which are the rest. Taking into account (4.9), $\tilde{\alpha}_{2,1}$, $\tilde{\alpha}_{3,1}$, $\tilde{\alpha}_{4,2}$, $\tilde{\alpha}_{4,3}$, $\tilde{\alpha}_{5,2}$, $\tilde{\alpha}_{5,3}$, $\tilde{\alpha}_{5,4}$, $\tilde{\alpha}_{6,5}$, $\tilde{\alpha}_{6,6}$, $\tilde{\alpha}_{6,7}$, $\tilde{\alpha}_{6,8}$ and $\tilde{\alpha}_{6,9}$ are simple invariants. The rest are compound invariants. An important quantity is the number of simple invariants at each order or type. We will denote it by $\hat{d}_i$. For $i = 1, \ldots, 6$ one finds,

$$\hat{d} \quad = \quad 0, \quad 1, \quad 1, \quad 2, \quad 3, \quad 5. \tag{4.17}$$

One of the most important aspects of this work is that it provides a geometrical interpretation of Vassiliev invariants in the sense that they are written as integrations of the type (4.2) and (4.4) and their generalizations. It would be interesting to study the relation between this integral representation and the one by Kontsevich [54]. Our approach is intrinsic to three dimensions and can be easily generalized to arbitrary three-manifold. In contrast, Kontsevitch's representation is basically two-dimensional (in the sense that the three-manifold is considered as a product $\mathbf{R} \times \mathbf{C}$) and therefore it is not obvious how to extend it to more general situations. However, for the case in which Kontsevich representation is defined, they should be related. This is supported by recent work [55] showing that Chern-Simons gauge theory in the Hamiltonian formalism leads to Kontsevich representation for Vassiliev invariants.

It is important at this point to discuss the relation between this work and the one by Bar-Natan in [56]. In [56] it is proved that the group factors of all Feynman diagrams with no collapsible propagators which contribute to a given order in g^2 (or x) can be regarded as Vassiliev invariants. These Vassiliev invariants have an entirely different origin than the knot invariants $\tilde{\alpha}_{ik}$. Bar-Natan's approach has two steps. First the observation that one can associate Feynman-like diagrams with no collapsible propagators to j-singular knots. Second that assigning the group factors of these diagrams to j-singular knots one constructs a set of rational numbers (weight system) that satisfies the axioms by Birman and Lin. Clearly, this observation is orthogonal to our results. It is important, however, to notice that the dimension of the space of Vassiliev invariants in [56] equals the number of independent group structures and therefore must have the same values as our d_i. Comparing the results in [56] with the values for d_i presented in (4.17) one finds complete agreement up to $i = 6$. We think that the calculation of these dimensions, in general, is more tractable in our approach. Recall that, as stated in the previous section, after equation (3.7) and its generalization for semisimple groups, the problem to compute d_i is reduced to the problem of finding c_i, or number of independent Casimirs of order i. The study of this issue is left for future work.

Taking into account the work [56] there appears a very appealing situation. The coefficient of x^i in the expansion of any quantum group invariant can be regarded as the inner product of two vectors of dimension d_i (the α_{ik} and the r_{ik}, $k = 1, \cdots, d_i$). Each of these two vectors is made out of Vassiliev invariants though each one has a different interpretation: while the α_{ik} are associated to the non-singular knot under study, the r_{ik} are in correspondence with a very precise set of j-singular knots, and associated to the group and representation under consideration.

5. Numerical knot invariants for all prime knots up to six crossings

In this section we will present the calculation of the knot invariants $\tilde{\alpha}_{ij}$ for all prime knots up to six crossings up to order six. Then we will show evidence supporting that there exist a normalization such that these knot invariants are integer-valued.

In order to compute the knot invariants $\tilde{\alpha}_{ij}(K)$ one could try to evaluate their integral expressions. This is certainly a long and tedious way to proceed. There is a faster way to carry out their computation using known information on the left hand side of (4.7). Indeed, the knot invariant $\langle W_K^R \rangle / \langle W_U^R \rangle$ is known for a variety of groups and representations for many knots on the three-manifold S^3. Taking its value for different cases one generates systems of linear equation where the unknowns are the $\tilde{\alpha}_{ij}(K)$. Recall that while the $r_{ij}(R)$ are group and representation dependent, they are knot independent. All dependence on K is contained in $\tilde{\alpha}_{ij}(K)$ which, on the other hand, are independent of the group and the representation. Up to order $i = 6$, which is the situation analyzed in this section, it is enough to consider the following cases: $SU(2)$ in an arbitrary representation of spin j (Jones and Akutsu-Wadati polynomials [20,21,24,14,18]), $SU(N)$ in the fundamental representation f (HOMFLY polynomial [22,21]), $SO(N)$ in the fundamental representation (Kauffman polynomial [23]), and $SU(2) \times SU(N)$ in representations of the form (j, f), *i.e.*, a representation of spin j in the subgroup $SU(2)$, and the fundamental in the subgroup $SU(N)$. These invariants are known and can be collected from the literature. They are listed in Appendix C.

The structure of the computation to be carried out is the following. Once the polynomial invariant corresponding to the left hand side of (4.7) is collected one replaces its variable q by e^x and expands in powers of x. For the case considered in this section one needs just the expansion up to order six. The coefficients of x^i are either polynomials in N, polynomials in j, or polynomials in N and j. On the other hand, on the right hand side of (4.7) the group factors are the ones in (3.11) and (3.12), which can be written explicitly using the values of the corresponding

Casimirs, which are listed in Appendix B. Again, one observes quickly that the group factors are polynomials in N, j, or both, N and j. Both sides of (4.7) must then be compared. This leads to series of linear equations which must be satisfied by the $\tilde{\alpha}_{ij}(K)$. It turns out that one encounters 5 equations for $\tilde{\alpha}_{2,1}(K)$, 5 equations for $\tilde{\alpha}_{3,1}(K)$, 12 equations for $\tilde{\alpha}_{4,1}(K)$, $\tilde{\alpha}_{4,2}(K)$ and $\tilde{\alpha}_{4,3}(K)$, 15 equations for $\tilde{\alpha}_{5,1}(K)$, ..., $\tilde{\alpha}_{5,4}(K)$, and 20 equations for $\tilde{\alpha}_{6,1}(K)$, ..., $\tilde{\alpha}_{6,9}(K)$. Those equations determine uniquely all the $\tilde{\alpha}_{ij}(K)$ up to order six.

The values of the $\tilde{\alpha}_{ij}(K)$ obtained in this way are, in general, rational numbers. For the trefoil, which will be labeled in the standard form 3_1 [57], one finds,

$$\begin{array}{ll}
\tilde{\alpha}_{2,1}(3_1) = 4, & \tilde{\alpha}_{6,1}(3_1) = \frac{32}{3}, \\
\tilde{\alpha}_{3,1}(3_1) = 8, & \tilde{\alpha}_{6,2}(3_1) = 32, \\
\tilde{\alpha}_{4,1}(3_1) = 8, & \tilde{\alpha}_{6,3}(3_1) = \frac{248}{3}, \\
\tilde{\alpha}_{4,2}(3_1) = \frac{62}{3}, & \tilde{\alpha}_{6,4}(3_1) = \frac{40}{3}, \\
\tilde{\alpha}_{4,3}(3_1) = \frac{10}{3}, & \tilde{\alpha}_{6,5}(3_1) = \frac{5071}{30}, \\
\tilde{\alpha}_{5,1}(3_1) = 32, & \tilde{\alpha}_{6,6}(3_1) = \frac{116}{30}, \\
\tilde{\alpha}_{5,2}(3_1) = \frac{176}{3}, & \tilde{\alpha}_{6,7}(3_1) = \frac{3062}{45}, \\
\tilde{\alpha}_{5,3}(3_1) = \frac{32}{3}, & \tilde{\alpha}_{6,8}(3_1) = \frac{17}{18}, \\
\tilde{\alpha}_{5,4}(3_1) = 8, & \tilde{\alpha}_{6,9}(3_1) = \frac{271}{30}.
\end{array} \tag{5.1}$$

Notice that these quantities satisfy the relations predicted in (4.9). We will not present the values of the $\tilde{\alpha}_{ij}$ for other knots since we are going first to normalize them properly.

As discussed in the previous section, these $\tilde{\alpha}_{ij}(K)$ are not universal in the sense that they depend on the group theoretical conventions used. It should be desirable to redefine them in such a way that they do not depend on those conventions. The simplest way to proceed would be to decide that these knot invariants take the value 1 for some knot in which none of them vanish. We will assume that this non-vanishing feature occurs for the simplest knot, the trefoil. It is certainly true up to order six as can be seen in (5.1). As we will show below, there is another normalization possibility, which is the one that we will finally take, in which all invariants up to order six seem to be integer-valued. Choosing the values for $\tilde{\alpha}_{ij}(3_1)$ as some selected integers, it turns out that the resulting invariants are integers. Our computations show that this happens to all prime knots up to six crossings. This leads us to conjecture that a similar picture holds for all orders and all knots.

Let us first redefine the universal knot invariants. We will denote them by $\tilde{\beta}_{ij}(K)$ as,

$$\tilde{\beta}_{ij}(K) = \frac{\tilde{\alpha}_{ij}(K)}{\tilde{\alpha}_{ij}(3_1)}. \tag{5.2}$$

These invariants are well defined if the quantities $\tilde{\alpha}_{ij}(3_1)$ do not vanish. We will assume this holds. On the other hand, it is clear from (5.2) that the invariants $\tilde{\beta}_{ij}(K)$ are independent of the normalization chosen for the Lie algebra generators. There is certainly a dependence on the basis for group factors which has been chosen but this dependence is intrinsic to their definition. For the right-handed trefoil, $\tilde{\beta}_{ij}(3_1) = 1$. In this normalization the algebra (4.9) simplifies. Instead of having numerical factors as the ones in (4.9) it turns out that just the product of two invariants, one of order i and another of order j, leads to the corresponding invariant of order ij.

Actually, it seems that there is a refined choice of the universal knot invariants which makes them integer-valued. This choice can only be observed after computing $\tilde{\beta}_{ij}(K)$ for many knots. We have carried out their calculation for all prime knots up to six crossings. The selected normalization up to order six is,

$$
\begin{aligned}
\beta_{2,1}(K) &= \frac{\tilde{\alpha}_{2,1}(K)}{\tilde{\alpha}_{2,1}(3_1)}, & \beta_{5,4}(K) &= \frac{\tilde{\alpha}_{5,4}(K)}{\tilde{\alpha}_{5,4}(3_1)}, \\
\beta_{3,1}(K) &= \frac{\tilde{\alpha}_{3,1}(K)}{\tilde{\alpha}_{3,1}(3_1)}, & \beta_{6,5}(K) &= 5071\frac{\tilde{\alpha}_{6,5}(K)}{\tilde{\alpha}_{6,5}(3_1)}, \\
\beta_{4,2}(K) &= 31\frac{\tilde{\alpha}_{4,2}(K)}{\tilde{\alpha}_{4,2}(3_1)}, & \beta_{6,6}(K) &= 29\frac{\tilde{\alpha}_{6,6}(K)}{\tilde{\alpha}_{6,6}(3_1)}, \\
\beta_{4,3}(K) &= 5\frac{\tilde{\alpha}_{4,3}(K)}{\tilde{\alpha}_{4,3}(3_1)}, & \beta_{6,7}(K) &= 1531\frac{\tilde{\alpha}_{6,7}(K)}{\tilde{\alpha}_{6,7}(3_1)}, \\
\beta_{5,2}(K) &= 11\frac{\tilde{\alpha}_{5,2}(K)}{\tilde{\alpha}_{5,2}(3_1)}, & \beta_{6,8}(K) &= 17\frac{\tilde{\alpha}_{6,8}(K)}{\tilde{\alpha}_{6,8}(3_1)}, \\
\beta_{5,3}(K) &= \frac{\tilde{\alpha}_{5,3}(K)}{\tilde{\alpha}_{5,3}(3_1)}, & \beta_{6,9}(K) &= 271\frac{\tilde{\alpha}_{6,9}(K)}{\tilde{\alpha}_{6,9}(3_1)}.
\end{aligned}
\tag{5.3}
$$

Notice that we have written only the simple knot invariants. The compound knot invariants are defined in such a way that they are products of simple ones:

$$
\begin{aligned}
\beta_{4,1} &= \beta_{2,1}^2, & \beta_{6,2} &= \beta_{3,1}^2, \\
\beta_{5,1} &= \beta_{2,1}\beta_{3,1}, & \beta_{6,3} &= \beta_{2,1}\beta_{4,2}, \\
\beta_{6,1} &= \beta_{2,1}^3, & \beta_{6,4} &= \beta_{2,1}\beta_{4,3}.
\end{aligned}
\tag{5.4}
$$

The explicit expressions of these universal simple knot invariants for all prime knots up to six crossings are presented in Table I. Knots have been labeled in their standard form [57].

knot	$\beta_{2,1}$	$\beta_{3,1}$	$\beta_{4,2}$	$\beta_{4,3}$	$\beta_{5,2}$	$\beta_{5,3}$	$\beta_{5,4}$	$\beta_{6,5}$	$\beta_{6,6}$	$\beta_{6,7}$	$\beta_{6,8}$	$\beta_{6,9}$
0_1	0	0	0	0	0	0	0	0	0	0	0	0
3_1	1	1	31	5	11	1	1	5071	29	1531	17	271
4_1	−1	0	17	7	0	0	0	−1231	71	−871	79	−271
5_1	3	5	261	39	157	14	13	123453	1247	34353	387	5853
5_2	2	3	134	22	69	6	7	45902	−42	14882	274	2702
6_1	−2	1	58	26	−19	−2	−3	−5582	442	−5042	686	−1742
6_2	−1	1	17	19	−13	−2	−3	2129	331	−931	463	−751
6_3	1	0	7	−7	0	0	0	511	209	−929	65	−449

Table I

The second row in Table I corresponds to the unknot, which has been labeled by 0_1. This, as well as the second row, which corresponds to the trefoil, can be thought as the choice of normalization. Once those are selected, the knot invariants β_{ij} for all other knots are fixed and are independent of the normalization of the Lie algebra generators. For knots which are chiral (not amphicheiral) we have selected in Table I the one which has $\beta_{3,1} > 0$. Notice that, as discussed before, for the two amphicheiral knots in the Table I, 4_1 and 6_3, one has $\beta_{ij} = 0$ for i odd. The values $\beta_{2,1}(3_1)$, $\beta_{2,1}(4_1)$ and $\beta_{2,1}(5_1)$ can be checked with the explicit computation of (4.2) in [27]. After taking into account the normalizations used in this work one finds full agreement. It is important to remark that the quantities $\beta_{ij}(K)$ are intrinsic to the knot, *i.e.*, as it follows from their construction they are framing independent.

In Table I only the simple knot invariants have been listed. Vassiliev invariants form an algebra [53] whose structure in the context of this work is represented by relations (5.4). The compound knot invariants are given by those relations from the values in Table I. The invariants constructed in this paper allow to build actuality tables using (4.10). There is one actuality table, or one Vassiliev invariant, associated to each β_{ij}, being this simple or compound. At a given type i, the space of Vassiliev invariants has dimension m_i, which at least up to $i = 7$ equals the quantity d_i [56]. On the other hand, we have a way to generate d_i sets of Vassiliev invariants or actuality tables. One could ask if the knot invariants constructed are independent or not, *i.e.*, if at each order they constitute a basis of Vassiliev invariants. Up to order 5 this holds. At order 6 we do not have enough information to come to a conclusion. The invariants should be computed for a higher number of knots. We would like, however, to conjecture that at each order i, the knot invariants proposed in this paper constitute a basis of Vassiliev invariants of type i.

The values of the Vassiliev invariants presented in Table I can be contrasted with the ones given in [46]. There, knots up to 7 crossings are considered, and Vassiliev invariants are presented up to order 4. It turns out that $\beta_{2,1}$ and $\beta_{3,1}$ are the same as in Table I. On the other hand, the three invariants of order 4 seem rather different. To compare two presentations of Vassiliev invariants of a given order one must take into account two facts. First, they might be presented in different basis; second to a Vassiliev invariant of order i one can substract a lower order Vassiliev invariant and still have a Vassiliev invariant of order i. In order to compare our results in Table I to Vassiliev's in [46] we must then ask the following question: there exist a non-singular 3×3 matrix A (which represent a change of basis) and a 3-vector b such that

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \beta_{4,1} \\ \beta_{4,2} \\ \beta_{4,3} \end{pmatrix} - \beta_{2,1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{5.5}$$

are the Vassiliev invariants of order 4 given in [46]? The answer is positive. From Table I and the results in [46] one constructs $3 \times 7 = 21$ linear equations whose unknowns are the 9 matrix elements of A and the 3 entries of b. Remarkably, the linear system has a solution:

$$A = \frac{1}{24} \begin{pmatrix} -12 & 1 & -2 \\ 0 & 0 & 1 \\ 12 & 0 & -2 \end{pmatrix}, \qquad b = \frac{1}{24} \begin{pmatrix} 9 \\ 5 \\ 2 \end{pmatrix}. \tag{5.6}$$

This fact is a highly non-trivial check which supports our construction.

6. Final remarks

In this paper we have presented new numerical knot invariants which generalize the one found in [27]. These invariants are defined from the universal form of the perturbative series expansion of a Wilson line in Chern-Simons gauge theory. One of the essential ingredients to be able to define this universal form of the series and therefore of the invariants is the identification of all diagrams which contribute to framing. This was done in [43]. In this way one works in the standard framing and defines invariants intrinsic to knots.

Besides being framing independent, the new knot invariants have very interesting features. First, they possess integral expressions and therefore have a geometrical origin. Second they are of finite type, or Vassiliev invariants, and there seems to exist a normalization in which they are integer-valued. Third, they are computable using information on polynomial invariants.

In this paper we have shown explicitly how the new numerical knot invariants are computed up to order 6. To carry out further the calculation of these numerical invariants one must first develop the analysis of independent group structures beyond order six. In sect. 3 we have presented an algorithm which computes these structures once the independent Casimirs are known. We need therefore to know the form of the independent Casimirs beyond order six. The characterization of the general form of these Casimirs is one of the open problems left for future work. Once the Casimirs are known there is still the problem of their calculation. For the representations and Lie algebras considered in this paper the calculation procedure described in appendix A can be applied to any order. For other representations and groups one could use similar techniques.

This work opens a variety of investigations. Certainly, the most important question that one would like to answer is if the infinite sequence of numerical invariants associated to each knot is a complete invariant, *i.e.*, if there are no inequivalent unoriented (invertible) knots such that their corresponding sequences are the same. In case this were true, one would like to know what is the minimum order (or type) such that invariants up to that order are able to distinguish all inequivalent unoriented knots. According to Table I in the previous section, for prime knots up to six crossings such a minimum order is 3.

Another important aspect which is worth to investigate is the relation of these knot invariants with Kontsevich's approach to Vassiliev invariants [54]. Both approaches should be basically the same in the situation in which Kontsevich's is defined. Though both are integral representations, our formulation is intrinsically three-dimensional. This opens the possibility to define integral representations for arbitrary three-manifolds. In this sense, one would like to search what is the situation for other manifolds and not just for S^3, which is the case considered in this paper. The new knot invariants might provide a powerful tool to define Vassiliev invariants in the general case. One would like also to ask if they are integer-valued on an arbitrary compact boundaryless manifold.

The new numerical knot invariants presented in this paper are introduced through the general power series expansion of (semisimple) group polynomial invariants. For particular groups these polynomial invariants satisfy skein relations. It would be very interesting to study if it is possible to write down a universal skein relation, valid for any semisimple group. If this were possible one would have a very useful tool to compute recursively the numerical knot invariants. Of course, to carry this out one needs first to define numerical invariants for links. This, on the other hand, could provide new insights on Vassiliev invariants for links, whose theory is much less developed than for the case of knots. We intend to pursue these investigations in future work.

Finally, one would like to know if the knot invariants corresponding to each order i are independent and therefore constitute a basis of Vassiliev invariants of type i. If this were true (as we conjecture) the space of quantum group invariants (at least with semisimple groups) is the same as the space of Vassiliev invariants. This would imply in particular that Vassiliev invariants would not be able to distinguish non-invertible knots, since Wilson lines are invariant under invertibility.

A necessary condition for this to hold is that the quantity d_i must coincide with m_i, the dimension of the space of Vassiliev invariants of type i. This has been proven to be true up to order 7 [56,53].

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APPENDIX A

In this appendix we present a summary of our group-theoretical conventions. We choose the generators of the Lie algebra A to be antihermitian such that

$$[T^a, T^b] = -f^{ab}{}_c T^c, \tag{A.1}$$

where f^{ab}_c are the structure constants. These satisfy the Jacobi identity,

$$f^{ab}{}_e f^{ec}{}_d + f^{cb}{}_e f^{ae}{}_d + f^{ac}{}_e f^{be}{}_d = 0 \tag{A.2}$$

The generators are normalized in such a way that for the fundamental representation,

$$\mathrm{Tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}, \tag{A.3}$$

where δ_{ab} is the Kronecker delta. This can always be done for compact semisimple Lie algebras which is the case considered in this paper.

The generators T^a in the adjoint representation coincide with the structure constants,

$$\big(T^a\big)_c{}^b = f^{ab}{}_c \qquad \text{(adjoint representation).} \tag{A.4}$$

The quadratic Casimir in the adjoint representation, C_A, is defined as

$$f^{ad}{}_c f^{bc}{}_d = C_A \delta^{ab}. \tag{A.5}$$

The value of C_A for the groups $SU(N)$ and $SO(N)$ is $-N$ and $-\frac{1}{2}(N-2)$ respectively. The Killing metric is chosen to be the identity matrix and therefore one can lower and raise group indices freely. For the case under consideration f_{abc} is totally antisymmetric.

The convention chosen in (A.1) seems unusual but it is the most convenient when the Wilson line is defined as in (2.13). If we had chosen if^{abc} instead of $-f^{abc}$, the exponential of the Wilson line would have had ig instead of g. Our convention also introduces a -1 in the vertex (see Fig. 1).

In the rest of this appendix we will describe how Casimirs are evaluated. For the fundamental representation they are computed by means of an algorithm presented in [58,59]. The evaluation procedure is simple. We think of the Casimir as a Feynman diagram, and introduce Feynman diagrammatic notation to replace the algebraic expressions. The first step is to get rid of the structure constants by means of the commutation relations *i.e.*, every factor like $f_{abc}T_c$ is substituted by $[T_b, T_a]$. Once this has been done the expression to be evaluated is a linear combination of traces of products of generators, with all their indices contracted. For example in the calculation of C_5 we get, among others, the following trace:

$$\mathrm{Tr}\big(T_a T_b T_c T_d T_b T_e T_d T_a T_e T_c\big). \tag{A.6}$$

In general, generators with the same index are not multiplied. If we write the matrix elements explicitly we can put together the pairs of generators with the same index, and therefore the quantities we are led to evaluate are of the form,

$$\Big(T^a\Big)_i^{\ j}\Big(T^a\Big)_k^{\ l}. \tag{A.7}$$

These group-theoretical objects are called projection operators. They are explicitly known for every classical Lie group except E_8 [58,59]. In the case of $SU(N)$ the projection operators are:

$$\Big(T^a\Big)_i^{\ j}\Big(T^a\Big)_k^{\ l} = -\frac{1}{2}\Big(\delta_i^{\ l}\delta_k^{\ j} - \frac{1}{N}\delta_i^{\ j}\delta_k^{\ l}\Big), \tag{A.8}$$

and in the case of $SO(N)$,

$$\Big(T^a\Big)_i^{\ j}\Big(T^a\Big)_k^{\ l} = -\frac{1}{4}\Big(\delta_k^{\ j}\delta_i^{\ l} - \delta^{jl}\delta_{ik}\Big). \tag{A.9}$$

Similar identities can be read from [58] for other groups. This solves the problem of calculating the Casimirs in the fundamental representation of these groups.

Higher representations can be introduced as properly symmetrized products of fundamental representations. These products span the representation ring of any compact Lie group. Extensions of (A.8) and (A.9) can be found, which would enable us to evaluate the analog of (A.7) in these more involved cases. Nevertheless in the case of $SU(2)$ in representation j nothing of this is needed since its rank is 1 and therefore all Casimirs are linear combinations of powers of the quadratic Casimir $j(j+1)$. The key identity is the product of two structure constants (which will be denoted by ε_{ijm}) with only one index contracted,

$$\varepsilon_{ijm}\varepsilon_{mkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{A.10}$$

Clearly this well-known identity simplifies the calculation of Casimirs in the case of $SU(2)$. Any Casimir gets reduced to the quadratic Casimir in the corresponding representation.

Using these rules we have computed all the Casimirs up to order six which have been used in this paper. These correspond to the fundamental representations of $SU(N)$ and $SO(N)$, and to an arbitrary spin j representation of $SU(2)$. Their values are contained in the following list:

$$SU(N)_f: \qquad \begin{aligned} C_2 &= -\frac{1}{2N}(N^2-1) \\ C_3 &= -\frac{1}{4}(N^2-1) \\ C_4 &= \frac{1}{16}(N^2-1)(N^2+2) \\ C_5 &= \frac{1}{32}N(N^2-1)(N^2+1) \\ C_6^1 &= \frac{1}{64}(N^2-1)(N^4+N^2+2) \\ C_6^2 &= \frac{1}{64}(N^2-1)(3N^2-2) \end{aligned} \tag{A.11}$$

$$
\begin{aligned}
SO(N)_f: \qquad C_2 =& -\frac{1}{4}(N-1)\\
C_3 =& -\frac{1}{16}(N-1)(N-2)\\
C_4 =& \frac{1}{256}(N-1)(N-2)(N^2-5N+10)\\
C_5 =& \frac{1}{1024}(N-1)(N-2)(N^3-7N^2+17N-10)\\
C_6^1 =& \frac{1}{4096}(N-1)(N-2)(N^2-7N+14)(N^2-2N+3)\\
C_6^2 =& \frac{1}{4096}(N-1)(N-2)(N-3)(7N-18)
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
SU(2)_j: \qquad C_2 =& -j(j+1)\\
C_3 =& -j(j+1)\\
C_4 =& 2j^2(j+1)^2\\
C_5 =& 3j^2(j+1)^2 - j(j+1)\\
C_6^1 =& 2j^3(j+1)^3 + 3j^2(j+1)^2 - 2j(j+1)\\
C_6^2 =& -2j^3(j+1)^3 + 5j^2(j+1)^2 - 2j(j+1)
\end{aligned}
\tag{A.13}
$$

APPENDIX B

In this appendix we present some elementary facts about semisimple Lie algebras relevant to the analysis of group factors in the perturbative expansion. Consider the part of a diagram depicted on the left of Fig. 13. It is possible to reduce its group factor by means of the totally antisymmetry of the structure constants f_{abc} at the cost of introducing C_A:

$$f_{abc}T_bT_c = \frac{1}{2}f_{abc}\big[T_b, T_c\big] = \frac{1}{2}f_{abc}f_{cbd}T_d = \frac{1}{2}C_AT_a. \qquad \text{(B.1)}$$

Actually, the Casimir C_A can be written in terms of C_2 and C_3, which is the expression that we will use wherever a "fishtail" appears:

$$C_3 = -\frac{1}{d(R)}f_{abc}\mathrm{Tr}(T_aT_bT_c) = -\frac{1}{2d(R)}C_A\mathrm{Tr}(T_aT_a) = -\frac{1}{2}C_AC_2 \quad \Rightarrow \quad C_A = -\frac{2C_3}{C_2}. \qquad \text{(B.2)}$$

There is a point which may be worth commenting. For an arbitrary semisimple Lie algebra $A = \oplus_{k=1}^{n}A_k$ in an arbitrary representation, the Wilson line can be imagined as consisting of n Wilson lines each one corresponding to one of the simple Lie algebras A_k in its respective representation. Therefore, a connected (sub)diagram can be regarded as a sum of similar subdiagrams where in each term the legs are attached to a component of the Wilson line and the sum runs over all these components. As a consequence one cannot identify the group factors of diagrams like those in Fig. 14 due to crossed terms in the product of the group factors of the subdiagrams. If the Lie algebra were simple their group factor would be the same due to the uniqueness of the Wilson line and to the reduction of the fishtails.

We call a subdiagram "separated" if its endpoints do not enclose any endpoints belonging to another subdiagram. For example, the diagrams displayed in Fig. 14 include only separated subdiagrams. In these cases the group factor is a product of the group factors of the subdiagrams.

Once a diagram has been drawn, its group factor is found in the following fashion. First one has to transform it into a sum of diagrams with only separated connected subdiagrams by means of the commutation relations. Take one of these new diagrams and consider one of its connected subdiagrams. Then reduce all the "fishtails" of the subdiagram chosen by means of (B.1) until one finishes with a Casimir-like subdiagram. The group factor of these Casimir-like subdiagrams has to be calculated separately and written in terms of its corresponding Casimirs by repeated use of the commutation relations. Then sum over all algebras A_k. This would give the contribution corresponding to the chosen subdiagram. Repeat this procedure for all subdiagrams and multiply the contributions of each subdiagram. The result is the group factor of the diagram we begun with. Following these steps for all diagrams present in the sum, we are done.

For example, the group factors of Fig. 14 are readily found to be

$$\begin{aligned} r(D_1) &= \Big(\sum_{k=1}^{n} \big(C_3^{(k)}\big)^2 \big(C_2^{(k)}\big)^{-1} \Big)^2 = r_{3,1}^2 , \\ r(D_2) &= \Big(\sum_{k=1}^{n} C_3^{(k)} \Big) \sum_{k=1}^{n} \big(C_3^{(k)}\big)^3 \big(C_2^{(k)}\big)^{-2} = r_{2,1} r_{4,2} . \end{aligned} \tag{B.3}$$

Notice that for a simple algebra these two factors would be the same. However, in the semisimple case they are different.

APPENDIX C

In this appendix we list the polynomial knot invariants which have been used in this paper to compute the numerical knot invariants up to order six for prime knots up to six crossings. Knot polynomials for the case of $SU(2)$ in a representation of spin j are collected from [18]. They are:

$$
\begin{aligned}
0_1: \quad & V_j = 1, \\
3_1: \quad & V_j = \frac{1}{[2j+1]} \sum_{l=0}^{2j} [2l+1](-1)^{2j-l} q^{-3(2j(2j+2)-l(l+1))/2}, \\
4_1: \quad & V_j = \frac{1}{[2j+1]} \sum_{l,k=0}^{2j} \sqrt{[2l+1][2k+1]}\, a_{kl} q^{l(l+1)-k(k+1)}, \\
5_1: \quad & V_j = \frac{1}{[2j+1]} \sum_{l=0}^{2j} [2l+1](-1)^{2j-l} q^{-5(2j(2j+2)-l(l+1))/2}, \\
5_2: \quad & V_j = \frac{1}{[2j+1]} \sum_{l,k=0}^{2j} \sqrt{[2l+1][2k+1]}\, a_{kl} (-1)^k q^{2j(2j+2)-l(l+1)+3k(k+1)/2}, \\
6_1: \quad & V_j = \frac{1}{[2j+1]} \sum_{l,k=0}^{2j} \sqrt{[2l+1][2k+1]}\, a_{lk} q^{l(l+1)-2k(k+1)}, \\
6_2: \quad & V_j = \frac{1}{[2j+1]} \sum_{i,l,k=0}^{2j} \sqrt{[2i+1][2k+1]}\, a_{li} a_{lk} (-1)^{2j-l-k} \\
& \qquad q^{-3(2j(2j+2)-k(k+1))/2-l(l+1)/2+i(i+1)}, \\
6_3: \quad & V_j = \frac{1}{[2j+1]} \sum_{r,s,l,k=0}^{2j} \sqrt{[2s+1][2k+1]}\, a_{kl} a_{lr} a_{rs} (-1)^{l+r} \\
& \qquad\qquad q^{-k(k+1)+s(s+1)+l(l+1)/2-r(r+1)/2},
\end{aligned}
\tag{C.1}
$$

where,

$$[m] = \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}, \tag{C.2}$$

is a q-number,

$$[m]! = [m][m-1][m-2]\dots[2][1], \tag{C.3}$$

corresponds to the q-factorial of a q-number, and,

$$a_{kl} = (-1)^{l+k-2j}\sqrt{[2k+1][2l+1]}\begin{pmatrix} j & j & k \\ j & j & l \end{pmatrix}, \tag{C.4}$$

is the duality matrix. This matrix is given in terms of the quantum Racah coefficients,

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{pmatrix} =& \Delta(j_1,j_2,j_{12})\Delta(j_3,j_4,j_{12})\Delta(j_1,j_4,j_{23})\Delta(j_3,j_2,j_{23}) \\ & \sum_{m\geq 0}(-1)^m[m+1]!\big\{[m-j_1-j_2-j_{12}]![m-j_3-j_4-j_{12}]! \\ & [m-j_1-j_4-j_{23}]![m-j_3-j_2-j_{23}]![j_1+j_2+j_3+j_4-m]! \\ & [j_1+j_3+j_{12}+j_{23}-m]![j_2+j_4+j_{12}+j_{23}-m]!\big\}^{-1},\end{aligned} \tag{C.5}$$

where the sum over m runs over all non-negative integers such that none of the q-factorials get a negative argument, and the symbol $\Delta(a,b,c)$ stands for

$$\Delta(a,b,c) = \sqrt{\frac{[-a+b+c]![a-b+c]![a+b-c]!}{[a+b+c+1]!}}. \tag{C.6}$$

For $SU(N)$ one has the following list of HOMFLY polynomial invariants [22,21,60]:

0_1 : $H = 1,$

3_1 : $H = \lambda\left(1+q^2-\lambda\,q^2\right),$

4_1 : $H = 1-\dfrac{1}{q}+\dfrac{1}{\lambda\,q}-q+\lambda\,q,$

5_1 : $H = \lambda^2\left(1+q^2-\lambda\,q^2+q^4-\lambda\,q^4\right),$

5_2 : $H = \lambda\left(1-q+\lambda\,q+q^2-\lambda\,q^2+\lambda\,q^3-\lambda^2\,q^3\right),$

6_1 : $H = \lambda^{-2}\,q^{-2}(1-\lambda+\lambda\,q-\lambda^2\,q-\lambda\,q^2+2\,\lambda^2\,q^2-\lambda^2\,q^3+\lambda^3\,q^3),$

6_2 : $H = \lambda^{-2}\,q^{-3}(1-\lambda-q+\lambda\,q+q^2-2\,\lambda\,q^2+\lambda^2\,q^2+\lambda\,q^3-\lambda\,q^4+\lambda^2\,q^4),$

6_3 : $H = 3-\dfrac{1}{\lambda}-\lambda+q^{-2}-\dfrac{1}{\lambda\,q^2}-\dfrac{1}{q}+\dfrac{1}{\lambda\,q}-q+\lambda\,q+q^2-\lambda\,q^2,$

where $\lambda = q^{N-1}$.

Finally, for the Kauffman polynomial invariants [23,60]:

$$
\begin{aligned}
3_1:\quad & R=2\alpha^2-\alpha^4+\left(-\alpha^3+\alpha^5\right)z+\left(\alpha^2-\alpha^4\right)z^2\\
4_1:\quad & R=-1+\alpha^{-2}+\alpha^2+\left(\frac{1}{\alpha}-\alpha\right)z+\left(-2+\alpha^{-2}+\alpha^2\right)z^2+\left(\frac{1}{\alpha}-\alpha\right)z^3\\
5_1:\quad & R=3\alpha^4-2\alpha^6+\left(-2\alpha^5+\alpha^7+\alpha^9\right)z+\left(4\alpha^4-3\alpha^6-\alpha^8\right)z^2+\left(-\alpha^5+\alpha^7\right)z^3\\
& \quad+\left(\alpha^4-\alpha^6\right)z^4\\
5_2:\quad & R=\alpha^2+\alpha^4-\alpha^6+\left(-2\alpha^5+2\alpha^7\right)z+\left(\alpha^2+\alpha^4-2\alpha^6\right)z^2+\left(\alpha^3-2\alpha^5+\alpha^7\right)z^3\\
& \quad+\left(\alpha^4-\alpha^6\right)z^4\\
6_1:\quad & R=\alpha^{-4}-\alpha^{-2}+\alpha^2+\left(\frac{2}{\alpha^3}-\frac{2}{\alpha}\right)z+\left(\frac{3}{\alpha^4}-\frac{4}{\alpha^2}+\alpha^2\right)z^2+\left(\frac{3}{\alpha^3}-\frac{2}{\alpha}-\alpha\right)z^3\\
& \quad+\left(1+\alpha^{-4}-\frac{2}{\alpha^2}\right)z^4+\left(\alpha^{-3}-\frac{1}{\alpha}\right)z^5\\
6_2:\quad & R=2+\alpha^{-4}-\frac{2}{\alpha^2}+\left(\alpha^{-5}-\alpha^{-3}\right)z+\left(3+\alpha^{-6}+\frac{2}{\alpha^4}-\frac{6}{\alpha^2}\right)z^2+\left(\frac{2}{\alpha^5}-\frac{2}{\alpha}\right)z^3\\
& \quad+\left(1+\frac{2}{\alpha^4}-\frac{3}{\alpha^2}\right)z^4+\left(\alpha^{-3}-\frac{1}{\alpha}\right)z^5\\
6_3:\quad & R=3-\alpha^{-2}-\alpha^2+\left(-\alpha^{-3}+\frac{2}{\alpha}-2\alpha+\alpha^3\right)z+\left(6-\frac{3}{\alpha^2}-3\alpha^2\right)z^2\\
& \quad+\left(-\alpha^{-3}+\frac{1}{\alpha}-\alpha+\alpha^3\right)z^3+\left(4-\frac{2}{\alpha^2}-2\alpha^2\right)z^4+\left(-\frac{1}{\alpha}+\alpha\right)z^5
\end{aligned}
$$

where $z=q^{\frac{1}{4}}-q^{-\frac{1}{4}}$ and $\alpha=q^{\frac{N-1}{4}}$. These invariant polynomials, as well as the rest of the invariants of this appendix, are given in the standard framing.

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